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Matematica per la Tecnologia - XXIII Ciclo

Stability in Convection Problems for Fluids and Flows in Porous Media with General Boundary Conditions

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Abstract

We investigate the onset of thermal convection in fluid layers and layers of fluids saturating a porous medium, when the temperature field is subject to the so called "natural" or Newton-Robin boundary conditions. Special attention is devoted to the limit case of fixed heat flux boundary conditions, corresponding to Neumann conditions on the temperature. Several interesting results, both from a physical and a mathematical point of view, appear when such conditions, coupled with one or more stabilizing fields (rotation, solute and magnetic field), are considered. The transition from fixed temperatures to fixed heat fluxes is shown to be destabilizing in all cases.

In Part I, we study fluids modeled by the Oberbeck-Boussinesq equations. When further fields are considered, additional equations and terms are included. In Part II, flows in porous media are described by the Darcy law, with and without the inclusion of an inertial term. Even for this system some stabilizing effects are considered.

In Chapter 3 we study the rotating Bénard problem. The wave number of the critical perturbations is shown to be zero, but only up to a *threshold of rotation speed* (depending on the kinetic boundary conditions).

In Chapter 4 the Bénard problem for a binary fluid is investigated. In this case, the stabilizing effect of a gradient of solute is *totally lost* for fixed heat fluxes. In Chapter 5 it is shown that, when the same system is rotating, the solute field is again stabilizing, and the critical wave number is positive in some regions of values of rotation and concentration gradient.

A more complex interaction, between rotation and magnetic field, is briefly discussed in Chapter 6. Again, zero and non-zero wave numbers

appear for different values of the parameters.

The solute Bénard problem is also investigated in Chapter 7, along the lines of the classical book of Chandrasekhar (1961), in a fully algebraic way. Here, finite slip boundary conditions, and Robin conditions on the solute field are also taken into account.

Flow in a rotating porous layer, and flow of a binary mixture in a porous layer, are studied in Chapters 8 and 9, with results qualitatively similar to those obtained for the Bénard system. The effects of inertia and rotation are investigated in Chapter 10, with a detailed analysis of the region in parameter space corresponding to the onset of stationary convection or overstability.

For most of the above systems, an asymptotic analysis for vanishing wave numbers was also performed, providing support to numerical calculations and some explicit analytic results.

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Chapter 1

Introduction

Landau and Lifshitz write in their book (Landau and Lifshitz, 1959) on Fluid Mechanics

> "*Yet not every solution of the equations of motion, even if it is exact, can actually occur in Nature. The flows that occur in Nature must not only obey the equations of fluid dynamics, but also be stable.*",

to underline the extreme relevance of the study of stability in the context of mathematical physics.

The object of this thesis is the study of stability/instability of fluid layers, both free fluids described by the Oberbeck-Boussinesq equations, and fluids saturating a porous medium, modeled by the Darcy law.

We show how *boundary conditions*, and in particular, *conditions on the temperature field, can have a dramatic effect on the onset of convection* in these systems. Other than simply produce a shift of the critical temperature gradient, some boundary conditions change in a qualitative way the convective motion of the fluid, or can compete with some stabilizing effects.

Here we recall first some basic results about convection in fluids and porous media.

1.1 Convection in fluids (classical results)

The space-time evolution of a motion of a Newtonian fluid is governed by the Navier-Stokes equations. One of the most widely used models in the theory of thermal convection - both for a homogeneous fluid and a mixture of fluids - is that given by the Oberbeck-Boussinesq equations. This last model is an approximation to the full thermo-mechanical equations which describe the motions of compressible viscous fluids.

From a mathematical point of view, both the Navier-Stokes (NS) equations and Oberbeck-Boussinesq (OB) equations are systems of partial differential equations which must be solved under suitable initial and/or boundary conditions.

They are non linear, and their well-posedness is still an open problem of fluid dynamics in the general case (see e.g. Flavin and Rionero (1996); Galdi (1994); Ladyzhenskaya (1969); Lions (1996); Temam (1979)). Well-posedness has been proved only for particularly symmetric cases, which can be reduced to two-dimensional problems (Galdi, 1994; Ladyzhenskaya, 1969; Temam, 1979). For the full three-dimensional problem, uniqueness has been proved for small initial data or for limited time intervals (Galdi, 1994; Ladyzhenskaya, 1969; Temam, 1979).

Determination of solutions is a still harder task, and only very rarely it is possible to find an explicit solution of these problems (Berker, 1963). Excluding these rare cases, if one wishes to study the characteristics of the solution one can either use approximate methods (e.g., numerical methods, asymptotic methods) or apply differential inequality techniques (qualitative analysis).

1.1.1 Oberbeck-Boussinesq equations

The Oberbeck-Boussinesq approximation to the full thermo-mechanical equations (the compressible Navier-Stokes equations) undoubtedly is the most widely used model of stratified fluids and thermal convection, (Chandrasekhar, 1961; Hills and Roberts, 1991; Joseph, 1976; Straughan, 2004). In the usual circumstance the temperature differences which are imposed at the boundary of a region of space occupied by a fluid (homogeneous fluid or a fluid mixture) may induce convection currents. Because of the presence of density gradients (induced by heating the fluid), the gravitational potential energy can be converted into motion through the action of buoyant forces. These forces appear when a fluid element in a gravity field is hotter (and then less dense) than its neighbor. There are many, physically important, situations in which the variations of density in a fluid are produced by variations in the temperature (and also in concentration of solute in the case of a mixture) of only moderate amounts and not by pressure differences. This happens e.g., in the case of water and in all convective motions of an *essential isochoric* kind (see Chandrasekhar (1961); Hills and Roberts (1991); Joseph (1976); Straughan (2004)). The NS equations for compressible fluids can be simplified considerably in these cases. The origin of the simplifications is due to the *smallness* of the coefficient of volume expansion: for water at 25 °C and 1.02 atmospheres of pressure, we have $\alpha = -(\frac{1}{\rho}\frac{\partial \rho}{\partial T})_p = 2.6 \times 10^{-4}\,{}^\circ\mathrm{C}^{-1}$ (here T is the absolute temperature and ρ is the density), for gases and other liquids $\alpha \in (10^{-4}, 10^{-3})\,{}^\circ\mathrm{C}^{-1}$. For variations in temperature not exceeding 10°, the variations in the density are at most 1 per cent. The same is for the variations in the other coefficients (kinematic viscosity, shear viscosity, specific heat). These *small* variations can, in general, be ignored. There is only one exception: the variability of ρ in the external force term in the momentum equation $\rho\,\mathbf{f}$, (see Chandrasekhar (1961)). Accordingly, we may treat ρ as a constant in all terms in the equations of motions except the one in the external forces; this is because the acceleration resulting from this force can be quite large. This is the Oberbeck-Boussinesq approximation. In this approximation the equation of state for the density is, (cf. Chandrasekhar (1961); Joseph (1976)):

$$\rho = \rho_0[1 - \alpha(T - T_0)], \tag{1.1}$$

where ρ_0 is the density at the reference temperature T_0 and α is the volume expansion coefficient (at a constant pressure):

$$\alpha = -\left(\frac{1}{\rho}\frac{\partial \rho}{\partial T}\right)_p .$$

With this state equation, the *Oberbeck-Boussinesq* equations for the basic flow are

$$\begin{cases} \mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla\left(\dfrac{p}{\rho_0}\right) + [1-\alpha(T-T_0)]\,\mathbf{g} + \nu\Delta\mathbf{v} \\ T_t + \mathbf{v}\cdot\nabla T = \kappa\Delta T & \text{in } \Omega\times(0,T^\star) \\ \nabla\cdot\mathbf{v} = 0 \end{cases} \tag{1.2}$$

where $\mathbf{v}$ is the velocity field, Ω is the domain of motion (typically a layer, see below), $\mathbf{g}$ is the external force per unit mass, typically gravity, ν is the kinematic viscosity, and κ is the coefficient of the thermometric conductivity, $T^\star \in (0,\infty]$.

To (1.2) we add the initial conditions

$$\mathbf{v}(\mathbf{x},0) = \mathbf{v}_0(\mathbf{x}), \quad T(\mathbf{x},0) = T_0(\mathbf{x}) \qquad \text{on } \Omega \tag{1.3}$$

and the boundary conditions

$$\mathbf{v}(\mathbf{x},t) = \mathbf{v}_\sigma(\mathbf{x},t), \quad T(\mathbf{x},t) = T_\sigma(\mathbf{x},t) \qquad \text{on } \partial\Omega\times[0,T^\star). \tag{1.4}$$

We note that we are interested in classical solutions to (1.2)-(1.4), i.e. $(\mathbf{v},T) \in C^2(\bar\Omega\times[0,T^\star))$ and then some compatibility conditions are assumed on the boundary and initial data.

1.1.2 Conduction solution of the Oberbeck-Boussinesq equations

Let $d > 0$ and $\Omega = \mathbb{R}^2 \times (-d/2, d/2)$. A stationary solution of (1.2) with boundary conditions $T(-d/2) = T_H$, $T(d/2) = T_L$, is given by the *conduction-solution*:

$$\begin{aligned} \bar{\mathbf{v}} = 0, \quad \bar{T} = T_a - \beta z, \quad \bar{\rho} = \rho_0(1 + \alpha\beta z), \\ \bar{p} = p_0 - g\rho_0(z + \tfrac{1}{2}\alpha\beta z^2) \end{aligned} \tag{1.5}$$

where $\beta = (T_H - T_L)/d$ is the gradient of the temperature field, $T_a = (T_H + T_L)/2$, and p_0 a constant.

We observe that this solution exists *for any value of the gradient* of the temperature. In the next sections we shall study the stability of this solution (the stability of the simple Bénard problem) and its stability in the presence of a rotation field, a solute field, and a magnetic field.

1.1.3 Stability of a Newtonian incompressible fluid

Let $m = (\mathbf{v}(\mathbf{x}, t),\ T(\mathbf{x}, t),\ p(\mathbf{x}, t))$ be a (sufficiently smooth) motion of a fluid $\mathcal{F}$, *i.e.*, a solution to the system (1.2)-(1.4) corresponding to given (regular) initial and boundary data $\mathbf{v}_0(\mathbf{x}), T_0(\mathbf{x})$ and $\mathbf{v}_\sigma, T_\sigma$, and external body force per unit mass $\mathbf{f}$. We call m a *basic motion* or *basic flow*.

Assume that the basic motion is perturbed at the initial instant ($t = 0$) in such a way that $\mathbf{v}$ and T are varied by a certain amount $\mathbf{u}_0, \vartheta_0$. Therefore the fluid will perform a new motion m' corresponding to the same boundary data and the same external forces as m, but with initial condition $\mathbf{v}_0 + \mathbf{u}_0, T_0 + \vartheta_0$. The motion m' will be given by $(\mathbf{v}(\mathbf{x}, t) + \mathbf{u}(\mathbf{x}, t), T(\mathbf{x}, t) + \vartheta(\mathbf{x}, t), p(\mathbf{x}, t) + \pi(\mathbf{x}, t))$ where $\mathbf{u}(\mathbf{x}, t)$, $\vartheta(\mathbf{x}, t)$ and $\pi(\mathbf{x}, t)$ are the perturbations to the kinetic, temperature and pressure fields, respectively. As the motion m' must satisfy the Oberbeck-Boussinesq equations, by subtraction with the basic motion, we obtain that the *perturbation* $(\mathbf{u}(\mathbf{x}, t), \vartheta(\mathbf{x}, t), \pi(\mathbf{x}, t))$ satisfies the following IBVP:

$$\begin{cases} \mathbf{u}_t + \mathbf{u} \cdot \nabla\mathbf{u} + \mathbf{v} \cdot \nabla\mathbf{u} + \mathbf{u} \cdot \nabla\mathbf{v} = -\nabla\pi - \alpha\vartheta\,\mathbf{g} + \nu\Delta\mathbf{u} & \text{in } \Omega \times (0, \infty) \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega \times (0, \infty) \\ \vartheta_t + \mathbf{v} \cdot \nabla\vartheta + \mathbf{u} \cdot \nabla T + \mathbf{u} \cdot \nabla\vartheta = \kappa\Delta\vartheta & \text{in } \Omega \times (0, \infty) \\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \vartheta(\mathbf{x}, 0) = \vartheta_0(\mathbf{x}) & \text{on } \Omega \\ \mathbf{u}(\mathbf{x}, t) = 0, \quad \vartheta(\mathbf{x}, t) = 0 & \text{on } \partial\Omega \times [0, \infty). \end{cases} \tag{1.6}$$

The stability problem of the basic flow m is then reduced to studying the time evolution of the perturbation $(\mathbf{u}(\mathbf{x},t), \vartheta(\mathbf{x},t)), \pi(\mathbf{x},t))$, *i.e.*, to studying the IBVP (1.6). In the sequel we shall suppose that problem (1.6) is *well posed* for any $t \geq 0$ and that the solution $\mathbf{u}(\mathbf{x},t)$ is classical, moreover we suppose that $\mathbf{u}_0, \vartheta_0 \in L^2(\Omega)$ and $\mathbf{u}(\mathbf{x},t), \vartheta(\mathbf{x},t) \in L^2(\Omega)), \forall t \geq 0$ and are bounded.

1.1.4 Definitions of energy-stability

According to the classical *energy method* (Lyapunov method) introduced by Serrin (1959) and Joseph (1965, 1966), we give the following definitions:

Definition 1.1. *The motion m is (energy)* ***stable*** *(with respect to perturbations in the initial data) if and only if*

$$\begin{array}{c} \forall \varepsilon > 0 \qquad \exists \delta(\varepsilon) > 0 : \\ \int_\Omega (u_0^2 + \vartheta_0^2)\, d\Omega < \delta \quad \Rightarrow \quad \sup_{t\in[0,\infty)} \int_\Omega (u^2(\mathbf{x},t) + \vartheta^2(\mathbf{x},t))\, d\Omega < \varepsilon. \end{array}$$

Definition 1.2. *The motion m is* ***unstable*** *if and only if it is not stable.*

Definition 1.3. *The motion m is* ***asymptotically stable*** *if and only if it is stable and moreover*

$$\begin{array}{c} \exists\, \gamma \in (0,\infty] : \\ \int_\Omega (u_0^2 + \vartheta_0^2)\, d\Omega < \gamma \quad \Rightarrow \quad \lim_{t\to\infty} \int_\Omega (u^2(\mathbf{x},t) + \vartheta^2(\mathbf{x},t))\, d\Omega = 0. \end{array}$$

If $\gamma \in \mathbb{R}$, then m is said ***conditionally*** *asymptotically stable. If $\gamma = \infty$, then m is* ***unconditionally*** *(or globally) asymptotically stable.*

From the previous definitions it appears that we can choose as Lyapunov function

$$V = \|\mathbf{u}(\mathbf{x},t)\|^2 + \|\vartheta(\mathbf{x},t)\|^2,$$

with $\|\cdot\|$ the $L^2(\Omega)$-norm.

1.1.5 The linearized system

From the definitions of the previous chapter it appears that the *rigorous* study of the stability of a basic flow is reduced to investigating - through of the energy-Lyapunov function - the time evolution of solutions to (1.6). However, because of the presence of the nonlinear term in (1.6), for a quite long time this research was a very difficult problem and only qualitative results were obtained in this sense, (see Kampe de Feriet (1949); Orr (1907); Reynolds (1895); Thomas (1943)).

Therefore, in order to simplify the problem and to obtain meaningful quantitative estimates, in the wake of work of Reynolds (1895) and Rayleigh (1916), the so-called *linearized instability method* has been used. Here, briefly recall the main points of this method. For a complete bibliography on this subject, see the monographs of Chandrasekhar (1961); Drazin and Reid (1981); Lin (1955).

Let $m_0 = (\mathbf{w}(\mathbf{x}), T(\mathbf{x}), p(\mathbf{x}))$ be a *stationary flow* of a viscous incompressible fluid $\mathcal{F}$ satisfying (1.2)-(1.4). Let

$$m' = (\mathbf{w}(\mathbf{x}) + \mathbf{u}(\mathbf{x}, t), T(\mathbf{x}) + \vartheta(\mathbf{x}, t)), p(\mathbf{x}) + \pi(\mathbf{x}, t))$$

another motion obtained by perturbing (at the initial instant) m_0. Then, as we have seen, the perturbation $(\mathbf{u}(\mathbf{x}, t), \vartheta(\mathbf{x}, t)), \pi(\mathbf{x}, t))$ satisfies the following system

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \mathbf{w}\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{w} = -\nabla\pi - \alpha\vartheta\,\mathbf{g} + \nu\Delta\mathbf{u} & \text{in } \Omega\times(0,\infty) \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega\times(0,\infty) \\ \vartheta_t + \mathbf{u}\cdot\nabla\vartheta + \mathbf{w}\cdot\nabla\vartheta + \mathbf{u}\cdot\nabla T = \kappa\Delta\vartheta & \text{in } \Omega\times(0,\infty) \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \quad \vartheta(\mathbf{x},0) = \vartheta_0(\mathbf{x}) & \text{on } \Omega \\ \mathbf{u}(\mathbf{x},t) = 0, \quad \vartheta(\mathbf{x},t) = 0 & \text{on } \partial\Omega\times[0,\infty). \end{cases} \tag{1.7}$$

Assume that $|\mathbf{u}|, |\nabla\mathbf{u}|$ and $|\nabla\vartheta|$ are *small* in such a way that we can neglect in (1.7) the nonlinear terms $\mathbf{u}\cdot\nabla\mathbf{u}$ and $\mathbf{u}\cdot\nabla\vartheta$. We thus obtain the linearized system

$$\begin{cases} \mathbf{u}_t + \mathbf{w}\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{w} = -\nabla\pi - \alpha\vartheta\mathbf{g} + \nu\Delta\mathbf{u} & \text{in } \Omega\times(0,\infty) \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega\times(0,\infty) \\ \vartheta_t + \mathbf{w}\cdot\nabla\vartheta + \mathbf{u}\cdot\nabla T = \kappa\Delta\vartheta & \text{in } \Omega\times(0,\infty) \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \quad \vartheta(\mathbf{x},0) = \vartheta_0(\mathbf{x}) & \text{on } \Omega \\ \mathbf{u}(\mathbf{x},t) = 0, \quad \vartheta(\mathbf{x},t) = 0 & \text{on } \partial\Omega\times[0,\infty). \end{cases} \tag{1.8}$$

System $(1.8)_{1-3}$ is linear and autonomous and therefore we may look for solutions of the following form (see e.g. Chandrasekhar (1961)):

$$\mathbf{u}(\mathbf{x},t) = e^{-\sigma t}\mathbf{u}_0(\mathbf{x}), \quad \pi(\mathbf{x},t) = e^{-\sigma t}\pi_0(\mathbf{x}), \quad \vartheta(\mathbf{x},t) = e^{-\sigma t}\vartheta_0(\mathbf{x}) \tag{1.9}$$

with σ *a priori* a complex number. Substituting (1.9) in (1.8), we have:

$$\begin{cases} -\nu\Delta\mathbf{u}_0 + \mathbf{w}\cdot\nabla\mathbf{u}_0 + \mathbf{u}_0\cdot\nabla\mathbf{w} + \nabla\pi_0 = \sigma\,\mathbf{u}_0 & \text{in } \Omega\times(0,\infty) \\ -\kappa\,\Delta\vartheta_0 + \mathbf{w}\cdot\nabla\vartheta_0 + \mathbf{u}\cdot\nabla T = \sigma\,\vartheta_0 & \text{in } \Omega\times(0,\infty) \\ \nabla\cdot\mathbf{u}_0 = 0 & \text{in } \Omega \\ \mathbf{u}_0(\mathbf{x}) = 0, \quad \vartheta_0(\mathbf{x}) = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.10}$$

The problem (1.10) is an eigenvalue problem in which σ and $(\mathbf{q}(\mathbf{x}) = (\mathbf{u}_0(\mathbf{x}), \vartheta_0(\mathbf{x}))$ are eigenvalue and eigenvector, respectively.

Assume that problem (1.10) admits a nonempty set Σ of eigenvalues σ and let $\mathbf{q}$ be a corresponding (4−dimensional) eigenvector satisfying (1.10). As the time evolution of the solutions (1.9) of linearized problem (1.8) depends on σ, the linear stability-instability problem is reduced to studying problem (1.10).

1.1.6 Definitions, the region which contains the eigenvalues

Definition 1.4. *A stationary motion m_0 is said to be linearly stable if and only if*

$$\exists\, k > 0 \textit{ such that} \quad Re(\sigma) \geq k \qquad \forall\sigma\in\Sigma,$$

m_0 is linearly unstable if and only if

$$\exists\, \sigma^\star \in \Sigma \quad \text{such that} \qquad Re(\sigma^\star) < 0,$$

where $\sigma = Re(\sigma) + i\, Im(\sigma)$, and i is the imaginary unity.

As far as the solvability of problem (1.10) is concerned, we observe that, if the domain Ω is bounded or if it is bounded at least in one direction and we restrict ourselves to eigenfunctions periodic along the directions in which Ω is unbounded (*normal modes*), it can be proved, as was shown in the isothermal case by Prodi (1962); Sattinger (1970) (see also Galdi and Rionero (1985); Straughan (2004)), that, when the solution $(\mathbf{w}, T, p)$ is sufficiently smooth, problem (1.10), set up in suitable function spaces, admits an at most countable number of eigenvalues $\{\sigma_n\}_{n\in\mathbb{N}}$. Precisely, let $\tilde{H}(\Omega) = \tilde{X}(\Omega) \times W^{1,2}(\Omega)$, where $\tilde{X}(\Omega)$ is the completion in the norm of $\tilde{L}_2(\Omega)$ of the space $\tilde{\mathcal{D}}(\Omega)$ of all solenoidal vectors with *complex* valued components which are in $C_0^\infty(\Omega)$, and $W^{1,2}(\Omega)$ is the usual Sobolev space, then the following theorem holds (see Prodi (1962); Sattinger (1970)).

Theorem 1.5. *Let Ω be a bounded domain of $\mathbb{R}^3$, $\partial\Omega \in C^2$, and let $\mathbf{w}, T \in C^1(\bar{\Omega})$. The eigenvalue problem (1.10), set up in a suitable subspace of $\tilde{H}(\Omega)$, admits a discrete set of eigenvalues $\Sigma = \{\sigma_n\}_{n\in\mathbb{N}}$ in the complex plane, each of finite multiplicity, which can cluster only at infinity. The eigenvalues lie in the parabolic region*

$$c_1[Im(\sigma)]^2 = Re(\sigma) + c_2$$

where c_1 and c_2 are some fixed positive constants. Moreover, the eigenvalues may be ordered in the following way

$$Re(\sigma_1) \le Re(\sigma_2) \le Re(\sigma_3) \le \cdots \le Re(\sigma_n) \le \cdots .$$

The corresponding eigenfunctions $\{\mathbf{q}_n\}_{n\in\mathbb{N}}$ are complete in $\tilde{H}(\Omega)$. That is, the class of all finite linear combinations of them is dense in $\tilde{H}(\Omega)$.

1.1.7 Critical parameter of linearized stability

As we have seen above, the problem of linear stability-instability is reduced to a study of the sign of $\mathrm{Re}(\sigma_1)$. Sometimes this is not a simple problem: for example the study of linear stability of the plane parallel Couette and Poiseuille flows is reduced to the classical Orr-Sommerfeld equation which is an ordinary differential equation with complex numbers coefficients which depend on independent variables. Therefore the solution is very complicated (see the monograph of Drazin and Reid (1981) for a complete examination) and a qualitative analysis or a numerical method must be used.

We also note, and this will be important in our study in this thesis, that the eigenvalues σ_i strongly depend on the boundary conditions on velocity and temperature. We will see in the next section the only (known) case in which an analytic solution can be found, that is the case of stress free conditions and fixed temperatures on both boundaries. For all other boundary conditions, numerical methods are required to evaluate eigenvalues and eigenfunctions of the problem.

Another difficulty arises when the domain is unbounded (in this case we cannot apply *a priori* Theorem 1.5). In the case when a domain is unbounded but bounded at least in one direction it is possible to consider perturbations which are periodic in the directions in which the domain is unbounded. Then the stability is studied in a *suitable cell* of periodicity which is a bounded domain. Although at first sight this hypothesis could appear restrictive, nevertheless the stability results that are reached in this way are, in many cases, in good agreement with the experiments, (cf. Chandrasekhar (1961); Joseph (1976)).

We notice that $\mathrm{Re}(\sigma_1)$ depend, in general, on the basic flow through a dimensionless positive parameter $\mathcal{P}$, say, such as Reynolds, Taylor, Rayleigh, or Chandrasekhar numbers, and, in the case of periodic perturbations, also on the associated wave number a (see below in the case of the Bénard problems). One wishes to find the least value $\mathcal{P}_c$ (*critical parameter*) of the parameter $\mathcal{P}$ for which $\mathrm{Re}(\sigma_1) = 0$, namely, the value of $\mathcal{P}$ at which instability sets in.

One also physically expects that $\mathcal{P} < \mathcal{P}_c$ for $\mathrm{Re}(\sigma_1) > 0$, while $\mathcal{P} > \mathcal{P}_c$ for $\mathrm{Re}(\sigma_1) < 0$. Though this seems reasonable, it is not always true and the dependence of $\mathrm{Re}(\sigma_1)$ on $\mathcal{P}$ must in principle be ascertained from case to case (cf. Drazin and Reid (1981), pp.11–12).

Sometimes, as we shall see in the case of the simple (or standard) Bénard problem with stress-free boundaries, it is easy to compute $\mathcal{P}_c$. Actually, let us rewrite (1.10) as

$$\mathfrak{L}\mathbf{q} = \sigma\mathbf{q} \tag{1.11}$$

where $\mathfrak{L}$ is a suitably defined linear operator which depends on $\mathcal{P}$ and, for periodic perturbations, also on the wave number a. Thus, in general, we have $\mathfrak{L} = \mathfrak{L}(\mathcal{P}, a)$.

Assume that the (strong) *principle of exchange of stabilities* (PES) holds, (Chandrasekhar, 1961):

$$\mathrm{Re}(\sigma_1) = 0 \qquad \Rightarrow \qquad \mathrm{Im}(\sigma_1) = 0 \tag{1.12}$$

namely, the *first* eigenvalue σ_1 is real at *criticality*. In this case (1.11) gives

$$\mathfrak{L}(\mathcal{P}, a)\mathbf{q}_1 = 0, \tag{1.13}$$

where $\mathbf{q}_1$ is the eigenvector corresponding to the eigenvalue σ_1. The meaning of this equation is the following: for each fixed a, equation (1.13) is an eigenvalue problem in $\mathcal{P}$. Solving this problem we get $\mathcal{P} = \mathcal{P}(a)$ and therefore the critical linear (in)stability parameter is given by

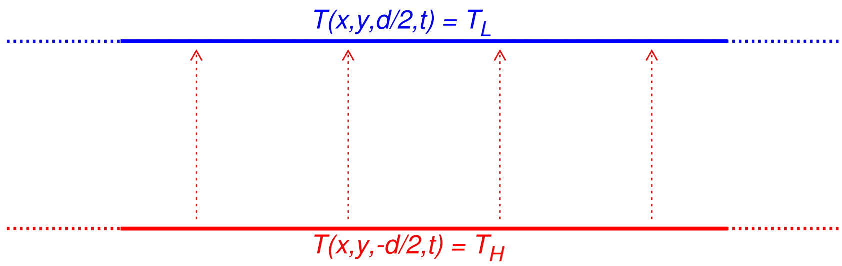
$$\mathcal{P}_c = \min_a \mathcal{P}(a).$$

We remark that, from the physical point of view, whenever the PES (1.12) holds, the instability sets in as a secondary stationary motion. As we shall see in the next chapters, there are many cases where (1.12) can be proved, and this happens, for instance, when the operator $\mathfrak{L}$ is symmetric or symmetrizable. However, there are also other cases, such as the Bénard problem with rotation (for Prandtl numbers less than 1) where (1.12) is violated. In these cases, at the onset of instability an oscillatory motion prevails: one says that one has a case of *overstability*.

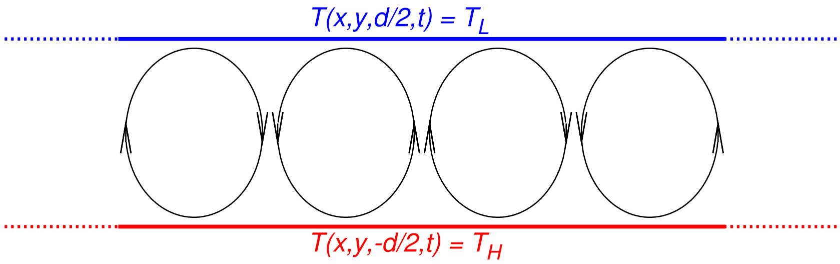
1.1.8 Linear instability of the Bénard problem

The problem of thermal stability-instability of the conduction solution of a horizontal layer of a fluid heated from below is known as *the Bénard problem* after the experiments of Bénard, (cf. Bénard (1900, 1901)).

Consider an infinite horizontal layer of viscous fluid in a gravity field, initially at rest, confined between two heat conducting planes. A constant adverse temperature gradient β is maintained between the walls by heating from below. The temperature gradient thus maintained is qualified as *adverse* since, on account of thermal expansion, the fluid at bottom is lighter than the fluid at the top; and this is a top-heavy arrangement which is potentially unstable.



If the temperature gradient is small, the fluid remains at rest and heat is transported through the fluid only by conduction (the conduction-solution to the OB equations). However, when the temperature gradient is increased beyond a certain critical value, the fluid undergoes time independent motions called *convection currents*. Heat is now transported through the fluid by convection as well as conduction. The experiments (see Bénard (1900, 1901); Koschmieder (1993); Rossby (1969)) show that the fluid arranges itself in a regular cellular pattern, and motion takes place only within the cells. The shape of the cells seems to depend strongly on the container, Koschmieder (1993). (In his famous experiment, Bénard found that the cells align themselves to form a regular hexagonal pattern).



A simple qualitative explanation of this phenomenon is the following. The fluid at the bottom expands because of the heating and becomes less dense than the fluid at the top. It therefore tends to rise. However, the fluid, being viscous, resists this buoyancy force. If temperature gradient β is small the viscous forces are dominant and the fluid remains at rest, heat being transported only by conduction. When a certain critical adverse temperature gradient β_c is exceeded, the buoyancy becomes large enough to overcome the viscosity of the fluid and the gravity, and convection begins.

The theoretical foundations for a correct interpretation of the foregoing facts were laid by Lord Rayleigh in a fundamental paper, Rayleigh (1916). Rayleigh showed that the correct non-dimensional parameter which decides the stability or instability is the number (now called *Rayleigh number*)

$$\mathcal{R}^2 = \frac{g\alpha\beta}{\kappa\nu} d^4$$

where $\mathcal{R}^2$ is the Rayleigh number, the other parameters have been introduced in section 1.1.1. Rayleigh showed that whenever

$$\mathcal{R}^2 > \mathcal{R}_c^2,$$

where $\mathcal{R}_c^2$ is the critical Rayleigh number, instability sets in and at the onset of convection a stationary secondary motion begins.

In the remaining part of this section we shall determine, in the case of stress-free boundaries, and for fixed temperatures at the boundaries, the critical Rayleigh number $\mathcal{R}_c^2$. For this, let us consider the OB equations and the conduction-solution given above. Let us consider a perturbed motion

$\bar{\mathbf{v}}+\mathbf{u}, \bar{T}+\vartheta, \bar{p}+\pi$, to the conduction solution (1.5). The perturbation $\mathbf{u}, \vartheta, \pi$ satisfies the system

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla\dfrac{\pi}{\rho_0} + \alpha g\vartheta\mathbf{k} + \nu\Delta\mathbf{u} \\ \vartheta_t + \mathbf{u}\cdot\nabla\vartheta = \beta w + \kappa\Delta\vartheta \qquad\qquad \text{in } \Omega\times(0,\infty), \\ \nabla\cdot\mathbf{u} = 0 \end{cases} \tag{1.14}$$

where $\mathbf{k}$ denotes the unit vector $(0,0,1)$. By introducing the non-dimensional variables (cf. Straughan (2004))

$$\begin{aligned} &t = t^\star\frac{d^2}{\nu}, \qquad p_1^\star P = \pi, \qquad P = \frac{\nu^2\rho_0}{d^2}, \qquad \Pr = \frac{\nu}{\kappa}, \\ &\mathbf{u} = \mathbf{u}^\star U, \qquad \vartheta = \vartheta^\star T^\sharp, \qquad T^\sharp = U\sqrt{d\frac{\beta\nu}{\kappa g\alpha}}, \qquad \mathcal{R} = \sqrt{\frac{\alpha g\beta d^4}{\kappa\nu}}, \\ &x = x^\star d,\ y = y^\star d,\ z = z^\star d, \qquad U = \frac{\nu}{d}, \end{aligned}$$

where Pr is the Prandtl number and $\mathbf{u} = (u, v, w)$, multiplying $(1.14)_1$ by d^3/ν^2, $(1.14)_2$ by $\sqrt{(d^6 g\alpha/\beta\nu^3 k)}$ and $(1.14)_3$ by ν, then the system (1.14) (dropping, as it is usual, the stars) becomes

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p_1 + \mathcal{R}\vartheta\mathbf{k} + \nu\Delta\mathbf{u} \\ \Pr(\vartheta_t + \mathbf{u}\cdot\nabla\vartheta) = \mathcal{R}w + \Delta\vartheta \qquad\qquad \text{in } \Omega_1\times(0,\infty), \\ \nabla\cdot\mathbf{u} = 0 \end{cases} \tag{1.15}$$

where $\Omega_1 = \mathbb{R}^2\times(-1/2, 1/2)$.

To this system we must add the boundary conditions which depend on the nature of the bounding surfaces. We shall distinguish two kinds of bounding surfaces, (Chandrasekhar, 1961): *rigid surfaces* on which no slip occurs and *stress-free surfaces* on which no tangential stresses act.

On rigid surfaces we require (cf. Chandrasekhar, 1961, pp. 21–22)

$$\mathbf{u} = 0, \qquad \vartheta = 0, \qquad\qquad \text{on } z = \pm 1/2, \tag{1.16}$$

on stress-free surfaces we require

$$w = 0, \quad \vartheta = 0, \quad u_z = v_z = 0 \qquad\qquad \text{on } z = \pm 1/2. \tag{1.17}$$

We assume that the perturbations are periodic functions in x and y of periods $2\pi/a_x$, $2\pi/a_y$, respectively, with $a_x > 0$, $a_y > 0$, and introduce the wave number $a = \sqrt{(a_x^2 + a_y^2)}$ and the periodicity cell $\Omega_0 = (0, 2\pi/a_x) \times (0, 2\pi/a_y) \times (0, 1)$.

The linearized version of (1.15) is

$$\begin{cases} \mathbf{u}_t = -\nabla(p_1) + \mathcal{R}\vartheta\mathbf{k} + \Delta\mathbf{u} \\ \mathrm{Pr}\vartheta_t = \mathcal{R}w + \Delta\vartheta & \text{in } \Omega_1 \times (0, \infty), \\ \nabla \cdot \mathbf{u} = 0 \end{cases} \tag{1.18}$$

By supposing that the perturbations have the forms:

$$\mathbf{u}(\mathbf{x}, t) = e^{-\sigma t}\mathbf{u}(\mathbf{x}), \qquad \vartheta(\mathbf{x}, t) = e^{-\sigma t}\vartheta(\mathbf{x}), \qquad p_1(\mathbf{x}, t) = e^{-\sigma t}p_1(\mathbf{x}),$$

then system (1.18) becomes

$$\begin{cases} -\sigma\mathbf{u} = -\nabla p_1 + \mathcal{R}\vartheta\mathbf{k} + \Delta\mathbf{u} \\ -\sigma\mathrm{Pr}\vartheta_t = \mathcal{R}w + \Delta\vartheta \\ \nabla \cdot \mathbf{u} = 0 \end{cases} \tag{1.19}$$

with boundary conditions (1.16) or (1.17) depending on the nature of the planes bounding the layer.

Now we prove that all the eigenvalues are *real*, and therefore the *principle of exchange of stabilities* is valid. Indeed, multiplying $(1.19)_1$ by $\bar{\mathbf{u}}$ and $(1.19)_2$ by $\bar{\vartheta}$ (where $\bar{\vartheta}$ denotes the complex conjugate of ϑ), integrating over Ω_0 and taking into account the solenoidality of the perturbation to the velocity field and the boundary conditions, it follows that

$$\begin{cases} -\sigma\|\mathbf{u}\|^2 = \mathcal{R}(\vartheta, w) - \|\nabla\mathbf{u}\|^2 \\ -\sigma\mathrm{Pr}\|\vartheta\|^2 = \mathcal{R}(w, \vartheta) - \|\nabla\vartheta\|^2, \end{cases}$$

where $(f, g) = \int_{\Omega_0} f\bar{g}\, d\Omega$ and $\|f\| = (\int_{\Omega_0} |f|^2\, d\Omega)^{1/2}$ denote the inner product and the norm in the space $\tilde{L}_2(\Omega)$. By adding the last two equations and by taking the imaginary part of both sides of the equation so deduced, we have

$$\mathrm{Im}(\sigma)(\|\mathbf{u}\|^2 + \mathrm{Pr}\|\vartheta\|^2) = 0$$

which implies $\text{Im}(\sigma) = 0$. Therefore, all the eigenvalues of (1.19) are real.

In order to eliminate ∇p_1 in (1.19), we take the *curl* of both sides of $(1.19)_1$, then we project on the z-axis. Moreover, we take the double *curl* of $(1.19)_1$ and the z-component of the equation so obtained. We easily get:

$$\begin{cases} -\sigma\zeta = \Delta\zeta \\ -\sigma\Delta w = \mathcal{R}\Delta^*\vartheta + \Delta\Delta w \\ -\sigma \text{Pr}\vartheta = \mathcal{R}w + \Delta\vartheta, \end{cases} \tag{1.20}$$

where $\zeta = \nabla \times \mathbf{u} \cdot \mathbf{k}$ is the third component of the vorticity and $\Delta^* = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

To system (1.20) we must add the boundary conditions

$$w = 0, \quad \vartheta = 0, \quad \zeta = 0, \quad w_z = 0 \qquad \text{on } z = \pm 1/2$$

for rigid-rigid boundaries, and

$$w = 0, \quad \vartheta = 0, \quad \zeta_z = 0, \quad w_{zz} = 0 \qquad \text{on } z = \pm 1/2$$

for stress-free boundaries.

Looking for solutions to (1.20) of *normal-modes* kind (see Chandrasekhar (1961)):

$$\begin{cases} \zeta = Z(z)\exp(i(a_x x + a_y y)) \\ w = W(z)\exp(i(a_x x + a_y y)) \\ \vartheta = \Theta(z)\exp(i(a_x x + a_y y)), \end{cases}$$

we have

$$\begin{cases} -\sigma Z = (D^2 - a^2)Z \\ -\sigma(D^2 - a^2)W = -\mathcal{R}a^2\Theta + (D^2 - a^2)^2 W \\ -\sigma \text{Pr}\Theta = \mathcal{R}W + (D^2 - a^2)\Theta, \end{cases} \tag{1.21}$$

where D denotes the derivative with respect to z. The boundary conditions become

$$W = 0, \quad \Theta = 0, \quad Z = 0, \quad DW = 0 \qquad \text{on} z = \pm 1/2$$

in the rigid-rigid case, and

$$W = 0, \quad \Theta = 0, \quad DZ = 0, \quad D^2W = 0 \qquad \text{on } z = \pm 1/2$$

for stress-free planes.

At criticality we have $\sigma = 0$, and therefore system (1.21) gives:

$$\begin{cases} (D^2 - a^2)Z = 0 \\ -\mathcal{R}a^2\Theta + (D^2 - a^2)^2W = 0 \\ \mathcal{R}W + (D^2 - a^2)\Theta = 0, \end{cases} \tag{1.22}$$

First equation of (1.22) and the boundary conditions show that, at the criticality, $Z(z) \equiv 0$. By eliminating Θ from the remaining equations we get

$$(D^2 - a^2)^3W = -\mathcal{R}^2a^2W. \tag{1.23}$$

The last equation, with boundary conditions

$$W = 0, \quad (D^2 - a^2)^2W = 0, \quad DW = 0 \qquad \text{on} z = \pm 1/2$$

in the rigid-rigid case, and

$$W = 0, \quad (D^2 - a^2)^2W = 0, \quad D^2W = 0 \qquad \text{on } z = \pm 1/2$$

in the free-free case, for any fixed a^2, can be considered as an eigenvalue problem for the Rayleigh number $\mathcal{R}^2$.

As we have said before, the critical value of linear stability-instability theory is obtained in this way: for any fixed a^2 we determine the minimum eigenvalue for $\mathcal{R}^2$; the minimum with respect to a^2 of the eigenvalue so obtained is the critical value we seek.

Now we shall solve (1.22) in the case of stress-free boundaries. Although this problem is not very useful for the laboratory experiments, nevertheless it is important from a theoretical point of view since it permits to find in a very simple analytical way the solutions.

In the stress - free case, assuming that $W(z) \in C^\infty$, from the boundary conditions and from equation (1.23) we easily obtain that

$$D^{(2m)}W = 0 \qquad \text{for } z = 0, \quad z = 1 \qquad \text{and } m = 1, 2, \ldots$$

From this it follows that

$$W = W_0 \sin n\pi z \qquad (n = 1, 2, \ldots),$$

where W_0 is a constant different from zero and n is a positive natural number. By substituting this solution in (1.23) we get the *characteristic equation*

$$\mathcal{R}^2 = \mathcal{R}^2(a^2, n^2) = \frac{(n^2\pi^2 + a^2)^3}{a^2}\,.$$

For any fixed a^2 the minimum $\mathcal{R}^2$ is obtained when $n = 1$, then

$$\mathcal{R}^2(a^2, 1) = \frac{(\pi^2 + a^2)^3}{a^2}\,.$$

Therefore the critical Rayleigh number of linear instability is given by

$$\mathcal{R}_c^2 = \min_{a^2>0} \frac{(\pi^2 + a^2)^3}{a^2} = \frac{27}{4}\pi^4 \simeq 657.511$$

and the corresponding wave number is $a_c = \pi/\sqrt{2} \simeq 2.221$.

When both the surfaces are rigid or one rigid and the other stress - free, by using a numerical analysis (cf., e.g., Straughan (2004), Appendix 2, or Chandrasekhar (1961), pp. 36–43), it can be found that

$$\mathcal{R}_c^2 \simeq 1707.76, \qquad a_c \simeq 3.117,$$

for rigid-rigid planes, and

$$\mathcal{R}_c^2 \simeq 1100.65, \qquad a_c \simeq 2.682,$$

for one rigid plane and the other stress-free.

1.1.9 The “insulating” case

When the simple Bénard system is subject to Neumann boundary condition on the temperature, some peculiar phenomena appear. By recalling the Fourier on heat propagation, this condition can be also called of *fixed heat flux.* This condition is not devoid of physical meaning, on the contrary, it is

	RR	RF	FF
$\mathcal{R}_c^2$	720	320	120

Table 1.1: Critical Rayleigh number of the Bénard system for fixed heat fluxes and combinations of Rigid and Free boundaries.

the correct boundary condition to employ when the the media surrounding the fluid has a very low conductivity (with respect to the fluid), as we discuss in Chapter 2. This conditions are equally appropriate to describe a surface subject to irradiation, and every case in which a constant flux of thermal energy enters (or exits) through a boundary.

It is known (see e.g. (Busse and Riahi, 1980; Chapman and Proctor, 1980)) that in this case the characteristic dimension of the convective cells tends to be infinite. This means that, in a practical experimental setup, the cells will tend to have the largest possible extension. Moreover, the critical Rayleigh number $\mathcal{R}_c^2$ can be explicitly calculated, and it tends to the integer values $720, 320, 120$, respectively for Rigid-Rigid, Rigid-Free, and Free-Free boundary conditions,

Most of this thesis is dedicated to investigate the effect of such boundary conditions when the system (Bénard system, or a flow in a porous medium) is subject to further physical, generally stabilizing, fields.

1.2 Stabilizing effects

In section 1.1.8 we recalled the main results on the onset of convection in the simple Bénard system. In many physical circumstances it is appropriate to include other fields and effects in the description of the system. Such effects are called *stabilizing* when they increase the critical temperature gradient needed to destabilize the conduction state.

A well known case is the rotating Bénard problem, which is relevant to many geophysical and industrial applications (e.g., crystal growing).

Another stabilizing effect is due to a solute dissolved in the fluid, in such a way that the solute density is decreasing from bottom to top in the layer.

Finally, when the fluid is electrically conducting, a magnetic field (directed along the vertical) is also stabilizing.

1.2.1 Rotation

The rotating Bénard problem attracted, in the past and increasingly today, the attention of many writers, (see Chandrasekhar (1961); Chandrasekhar and Elbert (1955); Flavin and Rionero (1996); Galdi and Padula (1990); Galdi and Straughan (1985); Kloeden and Wells (1983); Koschmieder (1967); Mulone and Rionero (1989); Nakagawa and Frenzen (1955); Niiler and Bisshopp (1965); Rossby (1969); Straughan (2004); Veronis (1959, 1966, 1968)). The stabilizing effect of rotation has been predicted by linear stability theory for any value of the Prandtl number Pr, (see Chandrasekhar (1961)), and has been confirmed by experiments, Koschmieder (1967); Nakagawa and Frenzen (1955); Rossby (1969).

For stress-free boundaries and fixed temperatures, and assuming the validity of PES (which can be proved for Pr>0.6766, Chandrasekhar (1961)), the critical Rayleigh number can be explicitly calculated as

$$\mathcal{R}_c^2 = \min_{a^2>0} \frac{(\pi^2+a^2)^3 + \pi^2\mathcal{T}^2}{a^2},$$

where $\mathcal{T}$ is the non dimensional number (Taylor number) related to the rotation speed $\hat{\Omega}$ by

$$\mathcal{T}^2 = \frac{4\hat{\Omega}^2 d^4}{\nu^2}.$$

The minimum of $\mathcal{R}^2$ is obtained for a critical wave number which is solution of the cubic

$$2(a^2)^3 + 3\pi^2(a^2)^2 - \pi^2(\pi^4+\mathcal{T}^2) = 0.$$

The Rayleigh number so obtained is an increasing function of $\mathcal{T}$, and the same can be proved for Pr<0.6766 when overstability is also possible.

1.2.2 Solute (and rotation)

The problem of a fluid layer heated and salted from below (the Bénard problem for a mixture) is studied for his relevance in geophysical applications, for example in the “salt pond” system, Tabor (1963); Tabor and R. (1965); Weinberger (1964).

The main effect of the solute is to induce a variation of density of the fluid. Even this density variation can be modeled as in (1.1), by adding a further term which depends on the solute concentration C, and, in a first approximation, on a reference density C_0 and a (positive) linear coefficient of density variation α_C,

$$\rho = \rho_0[1 - \alpha(T - T_0) + \alpha_C(C - C_0)].$$

In the OB approximation this expression introduce a further term in the body force. Transport of the solute inside the fluid should be also accounted for, by adding a new equation to the system

$$C_t + \mathbf{v} \cdot \nabla C = \kappa_C \Delta C,$$

with κ_C a diffusion coefficient. Note that this equation has the same form of $(1.2)_2$. In the basic motionless state, even this field present then a constant vertical gradient β_C (depending, in general, on the boundary conditions on the field). In a non dimensional form (see Section 4.1), the new system of equations for a perturbation to the base state depend on the solute Rayleigh number $\mathcal{C}$, and the Schmidt number $\mathrm{P_c}$

$$\mathcal{C}^2 = \frac{\alpha_C \beta_C g d^4}{\nu \kappa_C}, \qquad \mathrm{P_c} = \nu/\kappa_C.$$

At a difference from the simple Bénard system, overstability can now appear (see e.g. Joseph (1976)). For stress free kinetic boundary conditions, and for fixed temperatures, the effect of the solute field can be explicitly calculated in the linear case, giving a simple additive term (for stationary convection) to the critical Rayleigh number

$$\mathcal{R}_c^2 = \mathcal{R}_{c,B}^2 + \mathcal{C}^2$$

where $\mathcal{R}^2_{c,B} = 27\pi^4/4$ is the critical Rayleigh number in the absence of solute. Analytic formulas for the case of overstability can also be obtained ((Joseph, 1976).

It is also known, for fixed boundary temperatures (see e.g. Pearlstein (1981); Rionero and Mulone (1989)), that when the Bénard is subject to both rotation and a (stabilizing) solute gradient, both effects continue to be independently stabilizing. This is investigated in Chapter 5 for more general boundary conditions.

1.2.3 Magnetic field (and rotation)

When the fluid is an electrical conductor, an external magnetic field has also a stabilizing effect on the Bénard system. At a difference from the previous case (a solute field), when the system is also subject to rotation the total effect is not always stabilizing with respect to both fields. This phenomena are described, e.g., in Chandrasekhar (1961). In Chapter 6 this competition between the stabilizing effects is briefly discussed in the case of Newton-Robin and Neumann boundary conditions on the temperature.

1.3 Equations for flows in porous media

A porous medium is any matrix of a material that allows a flow of fluid (liquid or gas) through its interconnected internal voids, see e.g. Nield and Bejan (2006); Straughan (2008). Many materials exhibit such property, like sand, soil, concrete, various filters, hair, to name a few. Motion of a fluid inside such materials is clearly irregular, since the motion of the fluid particles is continuously influenced by the interaction with the medium.

Even if a microscopical description would be difficult, it is still possible to describe the system in terms of some *average quantities*, where the average is taken over a sufficiently large representative volume of material.

A typical quantity characterizing such media is the porosity, defined as the average fraction of volume which is void. This quantity can range from values close to 0 for very compact materials, like concrete, to large values for fiberglass or hair. Typical values for granular mixtures like sand or soil are in the range 0.4–0.6.

The average velocity of the fluid $\mathbf{v}$, *seepage velocity* or *filtration velocity* is the natural quantity to describe the motion of the fluid. The most elementary law for this velocity (or momentum) originates from Darcy (1856), and applies to a steady flow in a homogeneous and isotropic medium,

$$\mathbf{v} = \frac{K}{\mu}\nabla P$$

where ∇P is the gradient of pressure, K is a quantity depending on the microscopic geometry of the porous medium, called intrinsic permeability, and μ is the coefficient of dynamic viscosity of the fluid. Such law was empirically deduced by Darcy, but it has also been derived from general dynamic principles (see e.g. Rajagopal (2009); Straughan (2008) and references therein). Note that the Darcy law describes a steady state of flow, and this is generally appropriate in the description of the system. In some cases, however, transient behaviors of the fluid can be relevant, and the inclusion of some terms describing inertia of the fluid is necessary. A widely accepted form for this term is given by

$$\frac{\rho}{\varepsilon}\frac{\partial v}{\partial t},$$

where ρ is the fluid density, and ε is the porosity of the medium (see e.g. Vadasz (1998a)).

Even for this systems, it is possible to study thermal instability of a layer heated from below, and take into account the effect of stabilizing fields. The same considerations made for the Bénard system about an appropriate choice of thermal boundary conditions apply here. In several cases, Newton-Robin or fixed heat flux boundary conditions are required, and their effect is discussed in Chapters 8, 9, 10 of this thesis.

1.4 Numerical methods

The linear instability analysis performed on the various system described in this thesis follows the same general lines of the procedure described in Section 1.1.8, reducing finally the analysis to the solution of (ordinary) differential boundary value problems, similar to (1.21), together with its boundary conditions, that we rewrite here (for the case of rigid boundaries and fixed temperatures)

$$\begin{cases} -\sigma Z = (D^2 - a^2)Z \\ -\sigma(D^2 - a^2)W = -\mathcal{R}a^2\Theta + (D^2 - a^2)^2 W \\ -\sigma \mathrm{Pr}\Theta = \mathcal{R}W + (D^2 - a^2)\Theta, \end{cases} \tag{1.24}$$

$$\Theta = W = DW = (D^4 - a^2)^2 W = 0 \qquad \text{on } z = \pm 1/2.$$

These differential equations, and their boundary conditions, are homogeneous, and then for general values of the parameters ($\sigma, \mathcal{R}, a$ and Pr) only the identically null solution $Z = W = \Theta = 0$ exists. By fixing all parameters except one, we get an eigenvalue problem for the free parameter, which generally can not be solved analytically.

We study numerically such eigenvalue problems with a Chebyshev tau method, and solve the resulting algebraic generalized eigenvalue problem with the QZ algorithm according to the method described in Dongarra et al. (1996); Straughan (2004). The Newton-Robin thermal boundary condition, and the other homogeneous conditions on the solute density, velocity, and magnetic field, can be easily incorporated in the method.

The accuracy of the method has been checked by evaluation of the tau coefficients and, where possible, by comparison with known or analytical results. For an additional check, the simplified systems obtained when PES holds where also solved, in some cases, by shooting methods and compound matrix methods (Straughan, 2004). The compound matrix method is useful in the case of small values of a, since the algebraic eigenvalue problem obtained for the Chebyshev tau method becomes singular for $a \to 0$. In particular, the threshold values presented in section 3.6 were obtained with a compound matrix method.

Once solved the algebraic eigenvalue problem, further elaborations are necessary. When the PES is not assumed, as in the case of system (1.24), the quantity that is treated as eigenvalue is clearly σ, which in general is complex, with $\sigma = \text{Re}(\sigma) + i\text{Im}(\sigma) = r + is$. The eigenvalues so obtained are then sorted according to their real part r, as described in Section 1.1.6 for the spectrum of the full differential operator, obtaining in this way r_1. The Rayleigh number $\mathcal{R}$ is then be varied, until it is $r_1 = 0$, meaning that the system is at criticality and the Rayleigh number so obtained is the critical Rayleigh number. This procedure is necessarily done for a fixed values of all the other parameters involved, in particular the wave number a, so the critical Rayleigh number so found is really a $\mathcal{R}(a)$. It is then necessary to minimize $\mathcal{R}(a)$ with respect to a to finally obtain the $\mathcal{R}_c$ and a_c corresponding to the onset of convection.

Chapter 2

Boundary Conditions

2.1 Newton-Robin

The most common boundary conditions for the temperature are *thermostatic* boundary conditions. They are used, for example, in the classical works of Chandrasekhar (1961). The system we are investigating is a layer of fluid, bounded by the planes $z = \pm d/2$, and we can then suppose in this case that the value of the temperature is fixed at the boundaries

$$\begin{aligned} T &= T_H, \quad \text{on } z = -d/2 \\ T &= T_L, \quad \text{on } z = d/2, \end{aligned}$$

(here T_H and T_L denote a Higher and a Lower temperature).

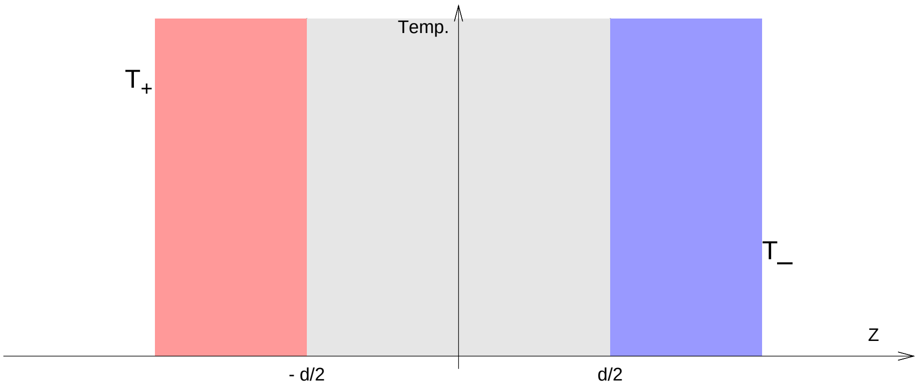
In general, boundary conditions for a physical quantity depend, just like equations, on the knowledge of the physical phenomena involved. In this case, boundary conditions on the temperature are physically justified by a description of heat transfer inside the media surrounding the fluid.

The case of a thermostat described above corresponds more closely to a solid body of high conductivity (with respect to the fluid) and high thermal capacity. But note that to maintain a fixed temperature, infinite conductivity and infinite thermal capacity of such media are ideally required.

In many *physically relevant* cases, however, the media surrounding the fluid are not (more of less ideal) thermostats (e.g. Sparrow et al. (1963)).

We can have, for example, a solid of relatively low conductivity, or a fluid conducting heat mainly by convection. In describing such situations we don't want, to a first approximation, to include these media in our system, but we just want to summarize the phenomena in new boundary conditions.

To describe a finite system we can suppose, for example, that the external media (low-conductivity solid, fluid, etc.) are, in turn, in contact with a thermostat, having fixed temperatures, say T_+, T_-.



For solids, heat transfer is described in the contest of the theory of thermal circuits, and we suppose that the heat flux is constant inside the solid bodies. For fluids we can use empiric laws, such us the *Newton's law of cooling* (Chapman and Proctor, 1980). This law is based on the assumption that the speed of cooling of a system depends linearly on the difference between the system temperature and an external temperature.

In both cases the heat flux Q at a boundary will be

$$Q \propto (T - T_{external}),$$

where $T_{external}$ denotes one of the temperatures T_+, T_-. We can then recall the Fourier law, which in one dimension is simply

$$Q \propto \frac{\partial T}{\partial z}.$$

From the previous expressions we have

$$\frac{\partial T}{\partial z} \propto (T - T_{external}),$$

or, say,

$$\frac{\partial T}{\partial z} = k_{rel}(T - T_{external}) \qquad \text{at a boundary.}$$

These boundary conditions are linear expressions in the normal derivative of the temperature at a boundary and the temperature itself. These kind of conditions, which can be regarded as a combination of Dirichlet and Neumann conditions, are generally called "Robin" in the mathematical literature.

In the case of temperature, because of the connection to the Newton law of cooling mentioned above, they are commonly referred to as *Newton-Robin boundary conditions*.

2.1.1 Preserving the basic solution

We are looking for motionless solutions of our system. The equation for the temperature inside the fluid is then (see eq. $(1.2)_2$)

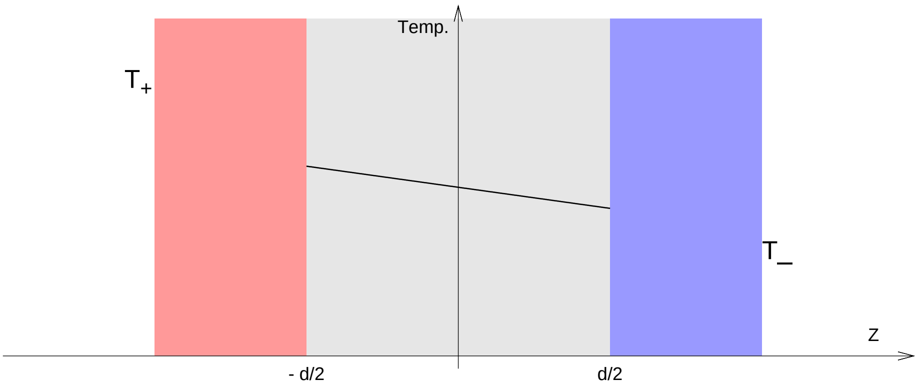
$$\Delta T(x, y, z) = 0, \quad x, y \in \mathbb{R}, z \in [-d/2, d/2],$$

subject to the most general Newton-Robin boundary conditions

$$\begin{aligned} a\frac{\partial T}{\partial z} + bT + c = 0, & \qquad z = -d/2, \\ a'\frac{\partial T}{\partial z} + b'T + c' = 0, & \qquad z = d/2, \end{aligned} \tag{2.1}$$

where all coefficients are constants (i.e. independent of the x, y coordinates). The only admissible solutions for this problem are linear functions of z

$$T = Az + B$$



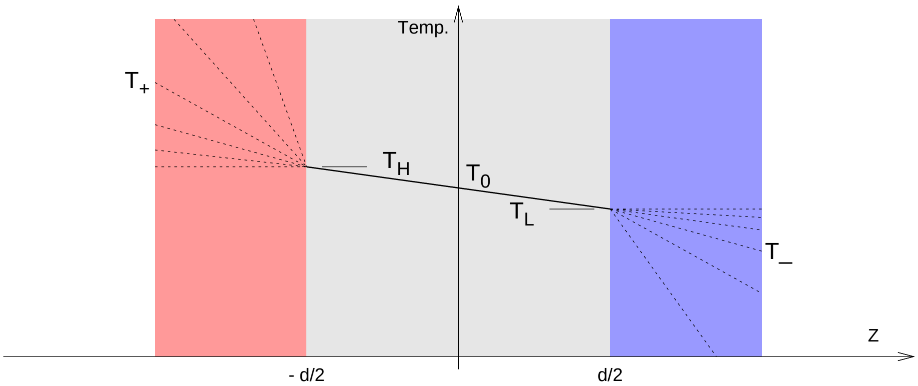
We can find the exact expression of $T(z)$, that is, determine the two unknowns A and B, using the two linear boundary conditions, since we have just two linear equations in two unknowns, but the general solutions for A, B are not simple functions of the constants a, b, c, a', b', c'. This implies also that the base solution is, in this way, affected by the boundary conditions.

Since we are just interested in the effect of Newton-Robin boundary conditions on the stability of the system, it is really not important how they affect the base solution, which is in any case a linear function of z. So it is preferable to reduce the degrees of freedom in the choice of the boundary conditions by imposing a fixed form of the base solution. Namely, we want to obtain a base solution with a given adverse gradient β and a given mean temperature T_0, that is

$$T = -\beta z + T_0.$$

By imposing this condition on both (2.1), taking also into account that expressions (2.1) are homogeneous in their three coefficients, we are left with just one free parameter for each of the two boundary conditions. We introduce then the two quantities $T_H = T_0 + \beta d/2$ and $T_L = T_0 - \beta d/2$, which are respectively an higher (H) and a lower (L) temperature, and the two free parameters $\alpha_H, \alpha_L \in [0, 1]$, and express our boundary conditions as

$$\begin{aligned} \alpha_H \left(\frac{\partial T}{\partial z} + \beta d \right) + (1 - \alpha_H)(T_H - T) = 0, &\quad \text{on } z = -d/2 \\ \alpha_L \left(\frac{\partial T}{\partial z} + \beta d \right) + (1 - \alpha_L)(T - T_L) = 0, &\quad \text{on } z = d/2. \end{aligned} \tag{2.2}$$



It would still be possible to express such boundary conditions in terms of external reference temperatures (like the temperatures T_+, T_- introduced above) and effective conductivity of the bounding media, but this would not have influence on the formulation of the problem and its solution, so in the remaining part of this thesis we will use the above form of the boundary conditions, without using explicitly any external reference temperature.

We will consider also the case of a boundary at which the *heat flux* is fixed. This case is generally described as a limit case of Newton-Robin boundary conditions for very low conductivity of the surrounding media, but it is also closely approximate by a surface subject to irradiation. Note that we can not obtain this kind of boundary condition directly from

$$T_z k_{rel}(T - T_{external}),$$

by sending the proportionality constant to zero. Imposing this, on one or both boundaries, leads to a base solution with a constant temperature.

2.1.2 Interpretation of the thermal boundary conditions

For $\alpha \in (0,1)$ conditions (2.2) are equivalent to the many Newton-Robin boundary conditions used in the literature, but, as we said, they ensure that the basic solution is always given by

$$\widehat{T}(x,y,z) = -\beta z + T_0,$$

this, in turn, ensures that the Rayleigh number is a quantity independent from the choice of boundary conditions, and allow us to discuss the net effect of boundary conditions on the solutions of our stability problem.

Conditions (2.2) include also the limit cases of fixed temperatures and fixed heat fluxes, respectively for $\alpha = 0$ and $\alpha = 1$.

2.2 Solute field

As is noted e.g. in Joseph (1976), boundary conditions for a solute dissolved in a fluid need not to be given just as fixed concentrations. If we suppose that solute migrates inside the media surrounding the fluid, which appears reasonable, by considerations similar to the case of temperature discussed above, we are lead (to a first approximation) to boundary conditions with a similar form, that is

$$\begin{aligned} &\gamma_H(C_z + \beta_C)d + (1-\gamma_H)(C_H - C) = 0, && \text{on } z = -d/2 \\ &\gamma_L(C_z + \beta_C)d + (1-\gamma_L)(C - C_L) = 0, && \text{on } z = d/2, \end{aligned}$$

where $\gamma_H, \gamma_L \in [0,1]$, β_C is a concentration gradient, and $C_H = C_0 + \beta_C d/2$, $C_L = C_0 - \beta_C d/2$ are respectively an higher (C_H) and lower (C_L) density, with C_0 a reference density. With this choice, the basic motionless solution has a fixed concentration gradient β_C, and fixed average C_0,

$$\widehat{C}(x,y,z) = -\beta_C z + C_0.$$

2.3 Finite slip

In Chapter 7 we study a system (solute Bénard system) subject to the above general boundary conditions on temperature and solute concentrations. We consider also a general expression of the kinetic boundary conditions, which include the stress-free and rigid conditions given by (1.17) and (1.16) as particular cases.

Rigid boundary conditions describe the interaction between a fluid and a solid boundary by prescribing that the fluid is at rest at the boundary (or, generally, it has the same velocity of the boundary). This condition is called also the *no-slip* boundary condition, and it is generally used in the description of fluid dynamic phenomena. In some cases, especially for gases, this condition is not verified when the geometry of the system has dimension of the order of the mean free path of the molecules. This effect appears even in liquids, as some recent results (Baudry and Charlaix, 2001; Craig et al., 2001; Priezjev et al., 2005; Webber, 2006) show.

In all cases, there appear to be a relation between tangential velocity at a boundary, and the shear strain of the fluid. Suppose the fluid is bounded from above by an horizontal surface Σ, then a horizontal component (say the component $\mathbf{u}_1 \equiv u$) of the velocity is given by

$$u|_\Sigma = -\lambda \epsilon_{1j} \mathbf{n}_j|_\Sigma$$

where ϵ_{ij} is the shear strain tensor, λ is a constant, $\mathbf{n}$ is the unit vector normal to the surface, given by $(0, 0, 1)$. The shear strain tensor is given by $\epsilon_{ij} = \partial_j \mathbf{u}_i + \partial_i \mathbf{u}_j$ and then we get

$$u = -\lambda \partial_x u.$$

When this boundary conditions are used, the velocity will still decrease at a boundary, but it will reach there a finite value. By extending the velocity profile inside the boundary, it can be seen the velocity goes (ideally) to zero at a distance exactly equal to λ inside the boundary, for any boundary value.

This parameter is for this reason called *slip length*. For incompressible fluids, it can be easily verified that this condition translates into

$$\partial_z w = \lambda \partial_z^2 w,$$

where w is the z component of the velocity (see also Chapter 7 for details).

Part I

Fluid layers

Chapter 3

Rotating Bénard system

3.1 Fluid equations

We consider an infinite layer $\Omega_d = \mathbb{R}^2 \times (-d/2, d/2)$ of thickness $d > 0$ filled with an incompressible homogeneous newtonian fluid $\mathcal{F}$, subject to the action of a vertical gravity field $\mathbf{g}$. We also assume that the fluid is uniformly rotating about the vertical axis z with an angular velocity $\widehat{\Omega}\mathbf{k}$, and denote by $Oxyz$ the cartesian frame of reference (with unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$) *rotating* about z with the same angular velocity $\widehat{\Omega}$. The equations of the fluid in the Boussinesq approximation are given by (see Chandrasekhar (1961)):

$$\begin{cases} \mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla \dfrac{p^\star}{\rho_0} + [1 - \alpha(T - T_0)]\mathbf{g} - 2\,\widehat{\Omega}\,\mathbf{k} \times \mathbf{v} + \nu \Delta \mathbf{v} \\ \nabla \cdot \mathbf{v} = 0 \\ T_t + \mathbf{v} \cdot \nabla T = k\,\Delta T \end{cases} \tag{3.1}$$

where $\mathbf{v}$, T, $p^\star$ are the velocity, temperature and pressure fields, respectively, and the field $p^\star$ includes the centrifugal force term. Further ρ_0, α, ν and k are positive constants which represent the density of the fluid at a reference temperature T_0, the coefficient of volume expansion, the kinematic viscosity and the thermometric conductivity, ∇ and Δ are the gradient and the Laplacian operators, respectively, and the suffix "t" denotes the partial time derivative.

For the velocity field, we assume that the boundaries are either rigid or stress free, and then

$$\begin{aligned} \mathbf{v} = 0, & \qquad \text{on rigid boundaries,} \\ \mathbf{k}\cdot\mathbf{v} = \partial_z(\mathbf{i}\cdot\mathbf{v}) = \partial_z(\mathbf{j}\cdot\mathbf{v}) = 0, & \qquad \text{on stress free boundaries.} \end{aligned} \tag{3.2}$$

Here we study the effect of more general boundary conditions on the temperature, besides the "thermostatic" boundary conditions used, for example, in the classical works of Chandrasekhar. In many physically relevant cases the media surrounding the fluid does not have the properties of a thermostat, as it could be, for example, a solid of relatively low conductivity, or a fluid conducting heat mainly by convection. The boundary conditions that can be derived for these systems are linear expressions in the normal derivative of the temperature at a boundary and the temperature itself, known as Newton-Robin boundary conditions.

Another case, which can be considered a limit case of Newton-Robin boundary conditions for very low conductivity of the surrounding media, is that of a boundary at which the *heat flux* is fixed (see e.g. Busse and Riahi (1980); Chapman and Proctor (1980)).

In the literature, many explicit forms of the Newton-Robin boundary conditions are used, but we find convenient to chose them in such a way that the basic solution is preserved:

$$\begin{aligned} \alpha_H(T_z+\beta)d + (1-\alpha_H)(T_H - T) = 0, & \qquad \text{on } z = -d/2 \\ \alpha_L(T_z+\beta)d + (1-\alpha_L)(T - T_L) = 0, & \qquad \text{on } z = d/2, \end{aligned} \tag{3.3}$$

where $\alpha_H, \alpha_L \in [0,1]$, $\beta > 0$, and $T_H = T_0 + \beta d/2$, $T_L = T_0 - \beta d/2$ are respectively an higher (T_H) and lower (T_L) temperature.

By choosing the values of α_H, α_L in the above expressions, we can obtain the various boundary conditions for the temperature cited above. In particular, if we choose $\alpha_H = \alpha_L = 0$ we obtain the *infinite conductivity* boundary condition, in which we fix the value of the temperature at the boundaries. For values of α_H or α_L belonging to the open interval $(0,1)$ we get the cases of finite conductivity at the corresponding boundary, or Newton-Robin conditions (Chapman and Proctor, 1980; Nield, 1964; Sparrow et al., 1963). For

$\alpha_H = 1$ and $\alpha_L = 1$ we get the *insulating* boundary conditions (Busse and Riahi, 1980; Clever and Busse, 1998), with a fixed heat flux $\mathbf{q}$ directed along the z axis at one or both boundaries, with $q = \beta k$.

The most important property of conditions (3.3) is that they *preserve the basic solution*, simplifying further analysis of the system. It can be easily verified, in fact, that problem (3.1) with the boundary conditions (3.2)-(3.3) and *any choice* of α_H and α_L has always a motionless solution, $\mathbf{v} = 0$, with the following expressions for the temperature field and the pressure field

$$\widehat{T}(x, y, z) = -\beta z + T_0, \quad p^* = p_0 + \rho_0 g(z + \frac{1}{2}\alpha\beta z^2), \tag{3.4}$$

with p_0 a real constant. This form of T implies also $\widehat{T}(x, y, -d/2) = T_H$, $\widehat{T}(x, y, d/2) = T_L$, meaning that T_H and T_L are the temperatures of the fluid in static conditions at the lower and upper boundaries, respectively. In the case $\alpha_H = \alpha_L = 1$, corresponding to a fixed heat flux at *both* boundaries, any function of the form $T(x, y, z) = -\beta z + C$, with C arbitrary real constant, is a solution. In this case we add to the system the following supplementary condition

$$\frac{1}{d}\int_{-d/2}^{d/2} \widehat{T}(x, y, z)\, dz = T_0$$

and obtain the same solution (3.4).

3.2 Equations of linear instability

A linear analysis of the stability of the motionless solution (3.4) of (3.1) can be done following the classical work of Chandrasekhar (1961), see also Straughan (2004). If we denote by $\mathbf{u} = (u, v, w)$, θ, p_1 the perturbations of the velocity, temperature and pressure fields, respectively, and by $\zeta = \mathbf{k}\cdot\nabla\times\mathbf{u}$ the z component of the vorticity field, we obtain the (linearized) system

$$\begin{cases} \Delta w_t = \mathcal{R}\Delta^*\theta - \mathcal{T}\zeta_z + \Delta\Delta w \\ \zeta_t = \mathcal{T} w_z + \Delta\zeta \\ Pr\,\theta_t = \mathcal{R} w + \Delta\theta, \end{cases} \tag{3.5}$$

where Δ^* is the two-dimensional Laplacian, the quantities $\mathcal{R}^2$, $\mathcal{T}^2$, and Pr are the Rayleigh, Taylor and Prandtl numbers, respectively

$$\mathcal{R}^2 = \frac{g\alpha\beta d^4}{\nu k}, \quad \mathcal{T}^2 = \frac{4\,\widehat{\Omega}^2 d^4}{\nu^2}, \quad \mathrm{Pr} = \frac{\nu}{k}\,.$$

As usual, we assume that the perturbation fields are sufficiently smooth and are periodic functions in the x and y directions. We denote by

$$\Omega_p = (0, 2\pi/a_x) \times (0, 2\pi/a_y) \times (-1/2, 1/2)$$

the periodicity cell, and by $a = (a_x^2 + a_y^2)^{1/2}$ the wave number (this is not a restriction as we can choose any other plane tiling pattern in the x, y directions, see Straughan (2004)). In the case of stress-free boundary conditions, in order to ensure uniqueness of the basic solution, we also require the "average velocity conditions" (Kloeden and Wells, 1983)

$$\int_{\Omega_p} u\, d\Omega_p = \int_{\Omega_p} v\, d\Omega_p = 0.$$

Following the standard analysis in normal modes of the system, we search then solutions of (3.5) in the form

$$\begin{cases} w = W(z)\exp\{\, i\,(a_x x + a_y y) + p\,t\} \\ \zeta = Z(z)\exp\{\, i\,(a_x x + a_y y) + p\,t\} \\ \theta = \Theta(z)\exp\{\, i\,(a_x x + a_y y) + p\,t\} \end{cases} \tag{3.6}$$

where $p = \sigma + i\tau$ is a complex constant, with $\mathrm{Re}(p) = \sigma$ and $\mathrm{Im}(p) = \tau$. Substituting expressions (3.6) into system (3.5) we obtain

$$\begin{cases} (D^2 - a^2)^2 W - \mathcal{T} DZ - \mathcal{R}\, a^2\, \Theta = p\,(D^2 - a^2)W \\ (D^2 - a^2)Z + \mathcal{T} DW = p\,Z \\ (D^2 - a^2)\Theta + \mathcal{R}\, W = p\,\mathrm{Pr}\;\Theta, \end{cases} \tag{3.7}$$

where "D" represents the operator of derivation along the z axis. In the absence of rotation, that is for $\mathcal{T} = 0$, field Z (that is linearly a stabilizing field) decouples from the other two fields. It is then possible to study the instability of the solutions through the following reduced system

$$\begin{cases} (D^2 - a^2)^2 W - \mathcal{R}\, a^2\, \Theta = p\,(D^2 - a^2)W \\ (D^2 - a^2)\Theta + \mathcal{R}\, W = p\,\mathrm{Pr}\;\Theta. \end{cases} \tag{3.8}$$

In these new variables, the hydrodynamic and thermal boundary conditions become

$$
\begin{array}{ll}
\text{on a rigid surface} & W = DW = Z = 0, \\
\text{on a stress-free surface} & W = D^2 W = DZ = 0, \\
\text{on } z = -1/2 & \alpha_H D\Theta - (1 - \alpha_H)\Theta = 0, \\
\text{on } z = 1/2 & \alpha_L D\Theta + (1 - \alpha_L)\Theta = 0.
\end{array}
\tag{3.9}
$$

We note that, in the case of FF boundaries, from $(3.7)_2$ it follows that $\sigma < 0$ or $\int_{-1/2}^{1/2} Z\,dz = 0$. System (3.7), subject to various combinations of the above boundary conditions, is an eigenproblem for the corresponding fields in which a, $\mathcal{R}$, $\mathcal{T}$, Pr, α_H and α_L are treated as parameters and p as the eigenvalue. From the definitions (3.6) we see that a perturbation, satisfying (3.7) or (3.8), will grow exponentially if $\sigma > 0$ for the corresponding eigenvalue.

If we denote by $p_1 = \sigma_1 + i\,\tau_1$ the eigenvalue with the largest real part in the spectrum of our problem (for the existence of such an eigenvalue see Straughan (2004)), *criticality* is obtained for $\sigma_1 = 0$. If at criticality it is also $\tau_1 = 0$ (and then $p_1 = 0$) we say that the principle of exchange of stabilities (PES) holds. If it is known (for example by analytical considerations, by previous calculations, or even by experimental data) that PES holds for some range of values of the parameters, then the study of (3.7) and (3.8) can be greatly simplified, since it is sufficient to consider the systems obtained for $p = 0$, which are also independent of Pr (see e.g. Chandrasekhar (1961)).

3.3 Sufficient conditions for PES

Theorem 3.1. *Let us consider system (3.8), which is obtained in the absence of rotation, with any combination of boundary conditions (3.9). It can be easily shown that the strong PES holds, i.e. all the eigenvalues are real numbers.*

Proof. We multiply eq. $(3.8)_1$ by $\overline{W}$ and eq. $(3.8)_2$ by $-\overline{\Theta}$ (where $\overline{A}$ denotes the complex conjugate of a field A) integrate over z on the interval $[-1/2, 1/2]$, and obtain

$$\begin{cases} \|(D^2 - a^2)W\|^2 - \mathcal{R}a^2(\Theta, W) + p\,(||DW||^2 + a^2\,||W||^2) = 0 \\ ||D\Theta||^2 + a^2\,||\Theta||^2 - \mathcal{R}(W, \Theta) + p\,\mathrm{Pr}\,||\Theta||^2 + S(\Theta) = 0, \end{cases}$$

where

$$S(\Theta) = \frac{1 - \alpha_H}{\alpha_H}|\Theta(-1/2)|^2 + \frac{1 - \alpha_L}{\alpha_L}|\Theta(1/2)|^2,$$

$||A||$ and $(A, B) = \int_{-1/2}^{1/2} A\overline{B}\,dz$ denote the usual norm and scalar product in $L^2([-1/2, 1/2])$. The term $S(\Theta)$, which is real (and non-negative), originates from the thermal boundary conditions $(3.9)_{3,4}$. If we multiply the second equation by a^2 and add it to the first, we obtain an equation whose imaginary part is

$$\tau(||DW||^2 + a^2\,||W||^2 + a^2\mathrm{Pr}\,||\Theta||^2) = 0\,.$$

It is then necessarily $\tau = 0$, i.e. all the eigenvalues p must be real. We conclude that, as is already well known for fixed boundary temperatures, for $\mathcal{T} = 0$ the strong PES holds. □

Now we consider system (3.7) (with $\mathcal{T} \neq 0$) and proceed as in Banerjee et al. (1985) and Chandrasekhar (1961). We multiply $(3.7)_1$ by $\overline{W}$, $(3.7)_2$ by $\overline{Z}$, $(3.7)_3$ by $-a^2\overline{\Theta}$, integrate over $[-1/2, 1/2]$ and sum. The imaginary part of the equation so obtained is

$$\tau(||DW||^2 + a^2\,||W||^2 + a^2\mathrm{Pr}\,||\Theta||^2 - ||Z||^2) = 0\,. \tag{3.10}$$

As before, Newton-Robin boundary conditions contribute with the real term $S(\Theta)$, which does not appear in the above equation. For any eigenvalue that has an imaginary part $\tau \neq 0$, (3.10) implies

$$||Z||^2 = ||DW||^2 + a^2\,||W||^2 + a^2\mathrm{Pr}\,||\Theta||^2\,. \tag{3.11}$$

Now we take the real part of the equation obtained multiplying $(3.7)_2$ by $\overline{Z}$

$$\sigma\,||Z||^2 = \mathcal{T}\mathrm{Re}(DW, Z) - (||DZ||^2 + a^2\,||Z||^2)$$

At criticality ($\sigma = 0$), we can construct the following chain of inequalities

$$\begin{aligned}
&||DZ||^2 + a^2\,||Z||^2 = -\mathcal{T}\mathrm{Re}(W, DZ) \leq \mathcal{T}\,||W||\,||DZ|| \leq \\
&\leq \frac{\mathcal{T}}{2}\left(\epsilon\,||W||^2 + \frac{1}{\epsilon}\,||DZ||^2\right) \leq \frac{\mathcal{T}}{2}\left(\frac{\epsilon}{\pi^2}\,||DW||^2 + \frac{1}{\epsilon}\,||DZ||^2\right) \leq \\
&\leq \frac{\mathcal{T}}{2}\left(\frac{\epsilon}{\pi^2}\,||Z||^2 + \frac{1}{\epsilon}\,||DZ||^2\right) \leq \frac{\mathcal{T}}{2}\left(\frac{\epsilon}{\pi^2 C^2} + \frac{1}{\epsilon}\right)||DZ||^2 ,
\end{aligned} \tag{3.12}$$

where ϵ is a positive number. The constant C^2 appearing in the last inequality is the Poincaré constant, $C^2 = \pi^2$ for FF and RR boundaries, $C^2 = \pi^2/4$ in the RF case. By choosing for ϵ the optimal value $\pi\,C$, from (3.12) and (3.11) we obtain

$$\left(1 - \frac{\mathcal{T}}{\pi C}\right)||DZ||^2 + a^2\,||DW||^2 + a^4\,||W||^2 + a^4\mathrm{Pr}\,||\Theta||^2 \leq 0\,, \tag{3.13}$$

We see that the previous equation cannot be satisfied if $\mathcal{T}^2 < \pi^2 C^2$, thus the following theorem holds.

Theorem 3.2. *For system (3.7) with thermal boundary conditions (3.9)*$_{3,4}$ *and any* α_H, α_L, *the principle of exchange of stabilities holds if* $\mathcal{T}^2 < \pi^4$ *for FF and RR hydrodynamic boundary conditions, or* $\mathcal{T}^2 < \pi^4/4$ *in the RF case.*

Remark 3.3. *We note that inequality (3.13) implies also the general validity of PES for large Prandtl numbers.*

Remark 3.4. *In the case FF, our numerical computations (see Sec. 3.7) show that the condition* $\mathcal{T}^2 > \pi^4$ *implies overstability, at least for Pr→ 0 and fixed heat fluxes. In this case, the condition given by Theorem 2 is optimal.*

3.4 Numerical methods

The solution of eigenproblems (3.7), (3.8) as it is well known, can be found analytically in terms of simple trigonometric functions in the case of free-free boundary conditions and fixed boundary temperatures (see Chandrasekhar

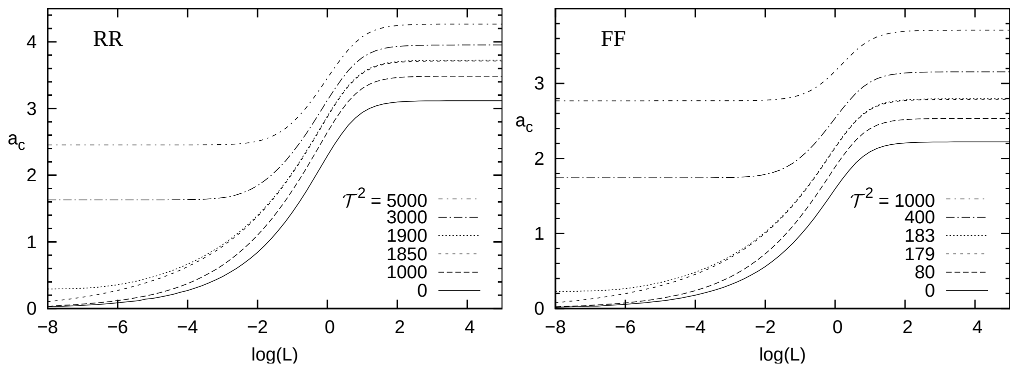


Figure 3.1: Values of a_c as a function of the parameter L for selected values of $\mathcal{T}^2$ for RR and FF boundary conditions. Notice the different behaviour for $L \to 0$ of the curves $\mathcal{T}^2 \leq 1850$ and $\mathcal{T}^2 \geq 1900$ for RR boundaries, and $\mathcal{T}^2 \leq 179$ and $\mathcal{T}^2 \geq 183$ for FF boundaries.

(1961); Straughan (2004)). For other boundary conditions there is no simple analytical solution, and the problems must be solved numerically.

We study numerically systems (3.7-3.8), with a Chebyshev tau method, and solve the resulting algebraic generalized eigenvalue problem with the QZ algorithm (Dongarra et al., 1996; Straughan, 2004).

3.5 Results for Newton-Robin boundary conditions

We consider first the case "finite" conductivity, that is boundary conditions $(3.9)_{3,4}$ for $\alpha_H, \alpha_L \in (0, 1)$. Even if different combination of values of α_H, α_L can be considered, we observed the most interesting effects for $\alpha_H, \alpha_L \to 1$, then for the sake of simplicity, we show here the cases of "symmetric" boundary conditions, that is $\alpha_H = \alpha_L \equiv \alpha$. For an easy comparison with previous works (e.g. Nield (1964); Sparrow et al. (1963)), we use in the following the quantity $L = (1 - \alpha)/\alpha$. In Fig. 3.1 we observe the asymptotic behaviour of the critical wave numbers for $L \to 0$ and $L \to \infty$, for RR and FF hydrodynamic boundary conditions. We see that, in both cases, $a_c \to 0$ as $L \to 0$,

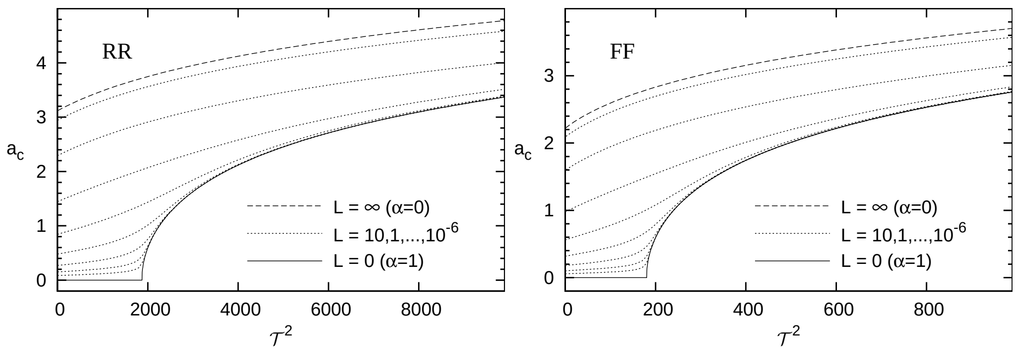


Figure 3.2: Values of a_c as a function of $\mathcal{T}^2$ for selected values of L ranging from $L = \infty$ (fixed temperatures) to $L = 0$ (fixed heat fluxes), RR and FF boundary conditions.

for a sufficiently small Taylor number. In the RR case, for example we see that a_c goes asymptotically to zero for $\mathcal{T}^2 \leq 1869$.

We investigate now more closely the behaviour of a_c for small values of L. As shown in Fig. 3.1, below some threshold value of $\mathcal{T}^2$, depending on the hydrodynamic boundary conditions, a_c seems to go to zero as $L \to 0$. To better understand this behaviour, it is useful to represent a_c as a function of $\mathcal{T}^2$ for a range of values of L, as we do in Fig. 3.2 for the case of RR and FF boundaries. The figure includes even the limit case $L = 0$, that is discussed more extensively in the next section. We see then clearly that $a_c \to 0$ as $L \to 0$ below a certain value of $\mathcal{T}^2$. The same qualitative behaviour is observed also for RF boundaries.

Fig. 3.3 represents the dependency of the Rayleigh number $\mathcal{R}^2$ on a, and underlines the big difference between the $L = 0$ and $L > 0$ cases. We see (in the limited range of the graphics) that for $L > 0$, $\mathcal{R}^2$ diverges to $+\infty$ for $a \to 0$, for any value of $\mathcal{T}^2$, and so clearly the minimum of $\mathcal{R}^2$ is always reached for some $a > 0$. On the other hand, for $L = 0$, $\mathcal{R}^2$ is always finite for $a = 0$, even when the minimum is reached for a positive value of a.

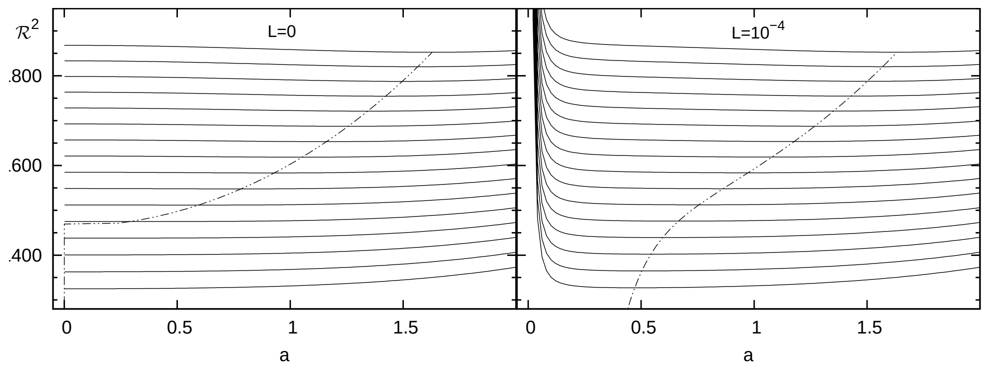


Figure 3.3: Values of $\mathcal{R}^2$ as a function of a for $L = 0$ and $L = 10^{-4}$ for RR boundary conditions. In both graphics, the continuous curves are computed for $\mathcal{T}^2 = 1500, 1600, \ldots, 3000$, from bottom to top. The dash-dotted lines are the loci of critical points $(a_c, \mathcal{R}_c^2)$, and so they intersect the continuous curves in their respective minima.

3.6 Results for fixed heat fluxes

We consider here the case $\alpha_H = \alpha_L = 1$ for the three combinations of surfaces RR, RF, FF, for moderate values of the Taylor number $\mathcal{T}^2$. It is known (Busse and Riahi, 1980; Chapman and Proctor, 1980) that in absence of rotation ($\mathcal{T}^2$= 0), when the thermal boundary conditions approach the "insulating" case, the critical wave number a_c tends to a minimum value equal to zero, and the corresponding critical Rayleigh numbers $\mathcal{R}_c^2$ tend to the *exact* integer values of $720, 320, 120$ for RR, RF, FF respectively. In Fig. 3.4 we show the dependency of the critical wave number a_c on $\mathcal{T}^2$, in the RR, RF, and FF cases, noting that the critical wave number departs from zero in all three cases above some value of $\mathcal{T}^2$.

The approximate threshold values of $\mathcal{T}^2$ at which a_c becomes greater than zero are given in Table 3.1.

Note that a_c becomes positive for small values of the Taylor number for both FF and RF boundary conditions and that a_c becomes positive first for the RF boundary conditions. $\mathcal{R}_c^2$ is an increasing function of $\mathcal{T}^2$, and a_c is increasing for $\mathcal{T}^2 > \mathcal{T}_a^2$. For $\mathcal{T}^2 = \mathcal{T}_a^2$ the dependency of the critical Rayleigh

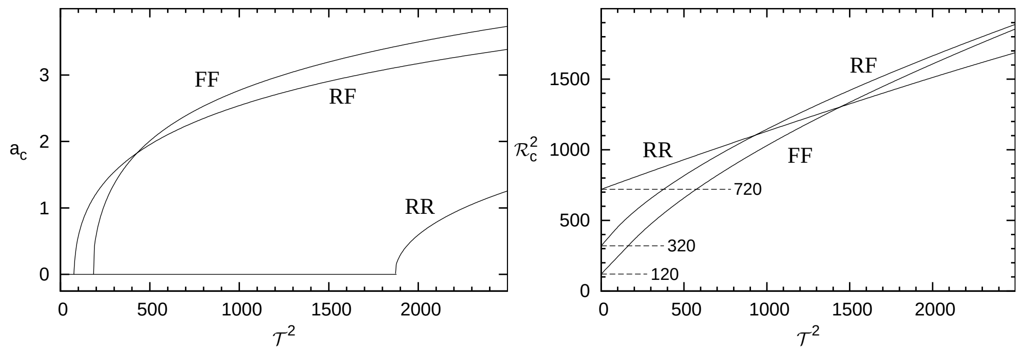


Figure 3.4: Critical wave number and Rayleigh number as a function of $\mathcal{T}^2$ for fixed heat fluxes at both boundaries. Note how the critical wave number is equal to zero only up to a certain threshold in all three cases.

Table 3.1: Approximate threshold values of $\mathcal{T}^2$ such that $a_c = 0$ for $\mathcal{T}^2 \lesssim \mathcal{T}_a^2$.

Boundaries	RF	FF	RR
$\mathcal{T}_a^2$	77.32	180.15	1868.86

number $\mathcal{R}_c^2$ on $\mathcal{T}^2$ remains regular (see right panel of Fig. 3.4).

3.7 Overstability

In Sec. 3.3 we have shown that for the rotating Bénard problem, PES holds for any Pr and for values of $\mathcal{T}^2 < \pi^4$ (RR and FF boundaries) or $\mathcal{T}^2 < \pi^4/4$ (RF boundaries), or for any $\mathcal{T}^2$ and large values of Pr.

The validity of PES in the general case ($a \geq 0$, $\mathcal{T}^2 \geq 0$, and any Pr) is an open problem. To complete our analysis, we have investigated with numerical methods the validity of PES for a wide range of values of $\mathcal{T}^2$ and Pr in the presence of Newton-Robin and "insulating" boundary conditions. Our computations confirm that, as expected, PES holds for sufficiently large Prandtl numbers for every combination of boundary conditions. Moreover, even for Pr$\to 0$, the limits for overstable convection derived in Sec. 3.3 are satisfied. In particular, we find that for fixed heat fluxes and Pr$\to 0$, PES

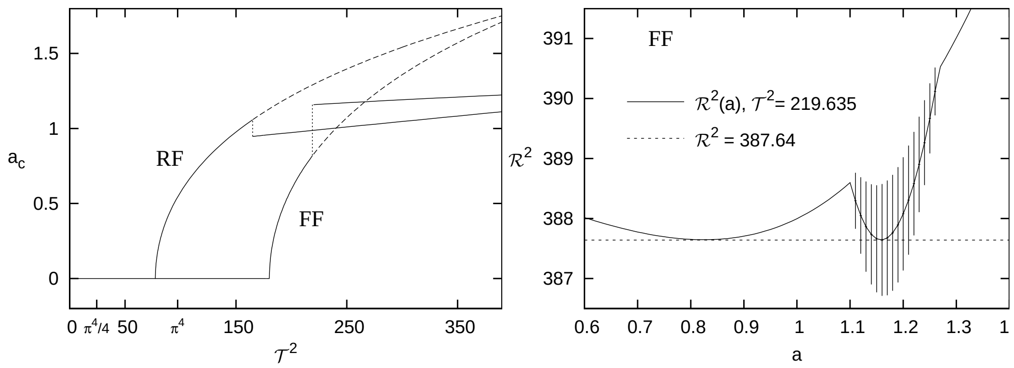


Figure 3.5: Left figure: the solid lines show the critical wave number as a function of $\mathcal{T}^2$ for fixed heat fluxes at both boundaries, RF and FF boundary conditions, and Prandtl number Pr = 0.025. The vertical dotted lines indicate a discontinuity in a_c, corresponding to the transition from simple convection to overstability. Right figure: curve $\mathcal{R}^2(a)$ for FF boundaries at $\mathcal{T}^2 = 219.635$, showing the transition between convective and overstable regime; the length of the vertical lines is equal to the imaginary part τ of the exponent p at the corresponding critical points.

holds for $\mathcal{T}^2$ less then $\approx 46.1, \approx 97.5, \approx 1654$, respectively for RF, FF, RR boundaries. We observe that for FF boundaries, the analytical threshold derived in Sec. 3.3 appears to be the best estimate, since the numerical value is very close to $\pi^4 \simeq 97.41$.

In Fig. 3.5 we show some results obtained for the physically relevant case of mercury (Pr= 0.025) in the limit case of fixed heat fluxes at both boundaries. We see, in the left panel, the transition from simple convection to overstability, corresponding to a sudden decrease of a_c for RF boundaries, and an increase in the FF case. The dashed lines correspond to the convective cases already presented in the previous section.

In the right figure we show a detail of the critical curve $\mathcal{R}^2(a)$ for the FF case at the transition between convection and overstability.

3.8 Main results

We have studied the rotating Bénard system subject to a variety of boundary conditions, with a particular attention toward thermal boundary conditions different from the common thermostatic conditions. For Newton-Robin boundary conditions, and, especially, in the limit case of fixed heat fluxes at both boundaries (the so-called *insulating* boundary conditions), the system stability exhibits a *qualitative* dependence on the Taylor number $\mathcal{T}^2$. We observe, in fact, that below a certain threshold of rotation speed the critical wave number tends to be zero, while, above that threshold, it is an increasing function of $\mathcal{T}^2$. This threshold should be experimentally observable and relevant even for boundary conditions *close* to the limit conditions of fixed heat fluxes, i.e. for poorly conducting boundaries. From a physical point of view, we expect that up to some threshold of rotation speed, the convection cells would probably be the largest allowed by the experimental setup, and their dimension should start to decrease when the rotation speed is increased above the threshold. A similar effect, in the case of Marangoni-Bénard convection, has already been observed (Mancho et al., 2002).

The nonlinear stability of the rotating Bénard problem for FF boundary conditions and fixed temperatures at the boundaries has been studied by many writers both with a weakly nonlinear method (Veronis, 1968) and Lyapunov functions (Flavin and Rionero, 1996; Mulone and Rionero, 1989, 1997; Straughan, 2004). The stabilizing effect of rotation on the onset of convection has been shown. A possible extension of our results is the study the nonlinear stability of the Bénard problem subject to the thermal boundary conditions considered here with a Lyapunov method by using also the new method introduced in Lombardo et al. (2008); Mulone and Straughan (2006).

Chapter 4

Solute field

In the simple Bénard problem the instability is driven by a density difference caused by a temperature difference between the lower and upper planes bounding the fluid. When the temperature gradient reaches a critical value the fluid gives rise to a regular pattern of motion (onset of convection).

If the fluid layer additionally has a *solute* dissolved in it, we have a binary fluid mixture and the phenomenon of convection which arises is called *double diffusive convection*. The study of stability and instability of motions of a binary fluid mixture heated and salted from below is relevant in many geophysical applications (Baines and Gill, 1969; Nield, 1967; Veronis, 1965) (see also Joseph (1976); Straughan (2004) and the references therein). It has been studied both in the linear and nonlinear case.

Here we consider the problem of a layer heated and salted from below. This means that in the motionless basic state we have a positive concentration gradient, having a stabilizing effect. The critical linear instability thresholds have been studied in the case of rigid and stress-free boundaries and for fixed temperatures and concentrations of mass. Here we consider more general boundary conditions on temperature and solute, in the form of Robin boundary conditions, which are linear expressions in the temperature (or solute concentration) and its normal derivative at a boundary. These boundary conditions are physically more realistic then simply fixing the value of the fields at a boundary, and they have a profound influence on the threshold of stability. A peculiarity of these boundary conditions for the temperature is that in the limit case of fixed heat fluxes the wavelength of the critical periodicity cell tends to infinity. In this work we investigate the influence of the solute field on this long-wavelength phenomenon, with the striking result that the critical parameters become *totally independent* from the solute field. This last aspect and non linear stability will require further analysis of the systems.

4.1 Bénard system for a binary mixture

We denote by $Oxyz$ the cartesian frame of reference, with unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, and we consider an infinite layer $\Omega_d = \mathbb{R}^2 \times (-d/2, d/2)$ of thickness $d > 0$ filled with a newtonian fluid $\mathcal{F}$, subject to a vertical gravity field $\mathbf{g} = -g\mathbf{k}$. We suppose that the density of the fluid depends on temperature T and on a solute concentration C according to the linear law $\rho_f = \rho_0[1 - \alpha_T(T - T_0) + \alpha_C(C - C_0)]$, where ρ_0, T_0 and C_0 are reference density, temperature and concentration, and α_T, α_C are (positive) density variation coefficients. In the Oberbeck-Boussinesq approximation, the equations governing the motion of the fluid are given by (see Joseph (1976))

$$\begin{cases} \mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla \dfrac{p}{\rho_0} + \dfrac{\rho_f}{\rho_0}\mathbf{g} + \nu \Delta \mathbf{v}, \\ \nabla \cdot \mathbf{v} = 0, \\ T_t + \mathbf{v} \cdot \nabla T = \kappa_T \Delta T, \\ C_t + \mathbf{v} \cdot \nabla C = \kappa_C \Delta C, \end{cases}$$

where $\mathbf{v}$ and p are the velocity and pressure fields. Further, ν and κ_T, κ_C are positive constants which represent kinematic viscosity and the thermal and solute diffusivity. ∇ and Δ are the gradient and the Laplacian operators, respectively, and the subscript "$_t$" denotes the partial time derivative.

For the velocity field, we assume that the boundaries are either rigid or stress free, and then

$$\begin{aligned} \mathbf{v} = 0, \qquad & \text{on rigid boundaries,} \\ \mathbf{k}\cdot\mathbf{v} = \partial_z(\mathbf{i}\cdot\mathbf{v}) = \partial_z(\mathbf{j}\cdot\mathbf{v}) = 0, \qquad & \text{on stress free boundaries.} \end{aligned}$$

For temperature we use Newton-Robin boundary conditions, which describe the physically relevant cases in which the media surrounding the fluid is not an ideal thermostat (see e.g. Joseph (1976)). In the literature, many explicit forms of the Newton-Robin boundary conditions are used, but we find convenient to choose them in such a way that, by varying a single coefficient, different thermal boundary conditions can be obtained, but the basic (motionless) solution is *preserved*:

$$\begin{aligned} \alpha_H(T_z + \beta_T)d + (1-\alpha_H)(T_H - T) = 0, \qquad & \text{on } z = -d/2 \\ \alpha_L(T_z + \beta_T)d + (1-\alpha_L)(T - T_L) = 0, \qquad & \text{on } z = d/2, \end{aligned} \tag{4.1}$$

where $\alpha_H, \alpha_L \in [0,1]$, $\beta_T > 0$, and $T_H = T_0 + \beta_T d/2$, $T_L = T_0 - \beta_T d/2$ are respectively an higher (T_H) and lower (T_L) temperature. For $\alpha_H, \alpha_L = 0$, we obtain the *infinite conductivity* boundary condition, in which we fix the value of the temperature at a boundary. For $\alpha_H, \alpha_L \in (0,1)$ we get the cases of finite conductivity at the corresponding boundary, or Newton-Robin conditions (Chapman and Proctor, 1980; Nield, 1964; Sparrow et al., 1963). For $\alpha_H, \alpha_L = 1$ we get the *insulating* boundary conditions (Busse and Riahi, 1980; Chapman and Proctor, 1980; Clever and Busse, 1998), with a fixed heat flux $\mathbf{q}$ directed along the z axis at one or both boundaries, with $q = \beta_T \kappa_T$.

For the solute field, by similar considerations (Joseph, 1976), we use boundary conditions depending on both concentration of solute and its normal derivative at the boundary surfaces. Again, to obtain a range of boundary conditions (from fixed concentrations to fixed fluxes of solute) depending on a single parameter on each boundary, while maintaining the basic solution, we use the following expressions

$$\begin{aligned} \gamma_H(C_z + \beta_C)d + (1-\gamma_H)(C_H - C) = 0, \qquad & \text{on } z = -d/2 \\ \gamma_L(C_z + \beta_C)d + (1-\gamma_L)(C - C_L) = 0, \qquad & \text{on } z = d/2, \end{aligned} \tag{4.2}$$

where $\gamma_H, \gamma_L \in [0,1]$, $\beta_C > 0$, and $C_H = C_0 + \beta_C d/2$, $C_L = C_0 - \beta_C d/2$ are respectively an higher (C_H) and lower (C_L) density.

The steady solution in whose stability we are interested is the motionless state, which, for any choice of the $\alpha_H, \alpha_L, \gamma_H, \gamma_L$ parameters, is

$$\mathbf{v} = \mathbf{0}, \quad \overline{T} = -\beta_T z + T_0, \quad \overline{C} = -\beta_C z + C_0. \tag{4.3}$$

Note that in (7.1) β_T, β_C are the temperature and concentration gradients.

The non-dimensional equations — here we use the non-dimensional form given in Joseph (1976), §56 — which govern the evolution of a disturbance to (7.1) are Joseph (1976)

$$\begin{cases} \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p_1 + (\mathcal{R}\vartheta - \mathcal{C}\gamma)\mathbf{k} + \Delta \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0, \\ \mathrm{P}_\theta(\vartheta_t + \mathbf{u} \cdot \nabla \vartheta) = \mathcal{R} w + \Delta \vartheta, \\ \mathrm{P_c}(\gamma_t + \mathbf{u} \cdot \nabla \gamma) = \mathcal{C} w + \Delta \gamma, \end{cases} \tag{4.4}$$

where $\mathbf{u} \equiv (u, v, w)$, ϑ, γ, p_1 are the perturbations to the velocity, temperature, concentration and pressure fields, respectively. The stability parameters in (4.4) are the Rayleigh numbers $\mathcal{R}, \mathcal{C}$ for heat and solute, and P_θ and $\mathrm{P_c}$ are the Prandtl and Schmidt numbers (as defined in Joseph (1976)), and are given respectively by

$$\mathcal{R}^2 = \frac{\alpha_T \beta_T g d^4}{\nu \kappa_T}, \qquad \mathcal{C}^2 = \frac{\alpha_C \beta_C g d^4}{\nu \kappa_C}. \tag{4.5}$$

Moreover

$$\mathrm{P}_\theta = \nu/\kappa_T \quad \text{and} \quad \mathrm{P_c} = \nu/\kappa_C$$

are the Prandtl and Schmidt numbers. Note that in (4.4) we have made use of the transformation $\mathcal{R}\vartheta = \vartheta_1, \mathcal{C}\gamma = \gamma_1$ and we have omitted the subscript "$_1$".

4.2 Linear instability equations

We perform a linear instability analysis of systems (4.4).

We follow the standard analysis of Chandrasekhar (1961), applying twice the curl operator to the first equation. We then consider only the linear terms of the resulting systems and obtain

$$\begin{cases} \Delta w_t = \mathcal{R}\Delta^* \vartheta - \mathcal{C}\Delta^* \gamma + \Delta\Delta w \\ \mathrm{P}_\theta \vartheta_t = \mathcal{R}w + \Delta\vartheta, \\ \mathrm{P_c}\gamma_t = \mathcal{C}w + \Delta\gamma. \end{cases} \tag{4.6}$$

where $\Delta^\star = \partial^2/\partial x^2 + \partial^2/\partial y^2$. We assume, as usual, that the perturbation fields are sufficiently smooth, and that they are periodic in the x and y directions (this is not a restriction, see Straughan (2004)). We denote by $a = (a_x^2 + a_y^2)^{1/2}$ the wave number. We search then solutions of both systems in the form

$$f = F(z)\exp\{\, i\,(a_x x + a_y y) + p\,t\} \tag{4.7}$$

for fields w, ϑ, γ, where $p = \sigma + i\tau$ is a complex constant. By substituting expressions (4.7) in (4.6) we obtain the following system for the perturbation fields W, Θ, Γ

$$\begin{cases} p\,(D^2 - a^2)W = (D^2 - a^2)^2 W + \mathcal{C}a^2\,\Gamma - \mathcal{R}a^2\,\Theta \\ p\,\mathrm{P}_\theta\,\Theta = (D^2 - a^2)\Theta + \mathcal{R}\,W, \\ p\,\mathrm{P_c}\,\Gamma = (D^2 - a^2)\Gamma + \mathcal{C}\,W, \end{cases}$$

where "D" represents the derivation along z. In this new variables, the hydrodynamic, thermal and solute boundary conditions become

on a rigid surface $\quad DW = W = 0,$

on a stress-free surface $\quad D^2 W = W = 0,$

on $z = -1/2 \quad \alpha_H D\Theta - (1-\alpha_H)\Theta = 0, \quad \gamma_H D\Gamma - (1-\gamma_H)\Gamma = 0,$

on $z = 1/2 \quad \alpha_L D\Theta + (1-\alpha_L)\Theta = 0, \quad \gamma_L D\Gamma + (1-\gamma_L)\Gamma = 0.$

When the principle of exchange of stabilities (PES) holds, a simplified form of both systems is obtained (Chandrasekhar, 1961).

4.3 Some results

It is well known (Busse and Riahi, 1980; Chapman and Proctor, 1980) that in the Bénard system, for Newton-Robin BCs approaching fixed heat fluxes on both boundaries, the critical wave number of the perturbations goes to zero (and so the wavelength goes to infinity), and the critical Rayleigh numbers $\mathcal{R}_c^2$ tend to the integer values $720, 320, 120$, respectively for RR, RF, and FF boundary conditions. This has been verified (Falsaperla et al., 2010a) also for a Darcy flow in a porous medium, with $\mathcal{R}_c^2 = 12$. We check here how a solute field affects the stability of a fluid under Newton-Robin or fixed heat flux BCs.

In general, the eigenvalue problems obtained for this kind of boundary conditions must be solved numerically. We solved our eigenvalue problems with a Chebyshev Tau method (see Straughan (2004) and Dongarra et al. (1996)). The accuracy of the method has been checked by evaluation of the tau coefficients, by comparison with known or analytical results, and, when PES holds, comparing the solutions of PES and non-PES problems.

We performed a series of computations for different choices of Prandtl and Schmidt numbers, and thermal and hydrodynamic BCs. In the following we present some results for stress free boundaries. In this case the analytic solution is known (see e.g. Lombardo and Mulone (2002)) for thermostatic boundaries, and overstability phenomena are present for $\mathrm{P_c}/\mathrm{P}_\theta > 1$. We show graphics obtained for $\mathrm{P_c} = 3$ and $\mathrm{P}_\theta = 1$. The same qualitative behavior appears also for more physically meaningful values, such as $\mathrm{P_c} = 670$ and $\mathrm{P}_\theta = 6.7$ (for sea water).

We see in Fig. 4.1 that the overstability region disappears during the transition from fixed temperatures to fixed heat fluxes, (at a smaller value of α for sea water). The solute, as expected, has a stabilizing effect, but we note that in the limit case of fixed heat fluxes the stabilizing effect is **totally lost**, since the critical Rayleigh number becomes *independent of the concentration gradient*. This result is somehow surprising, and will require further investigation. (In the case of the rotating Bénard system we observed

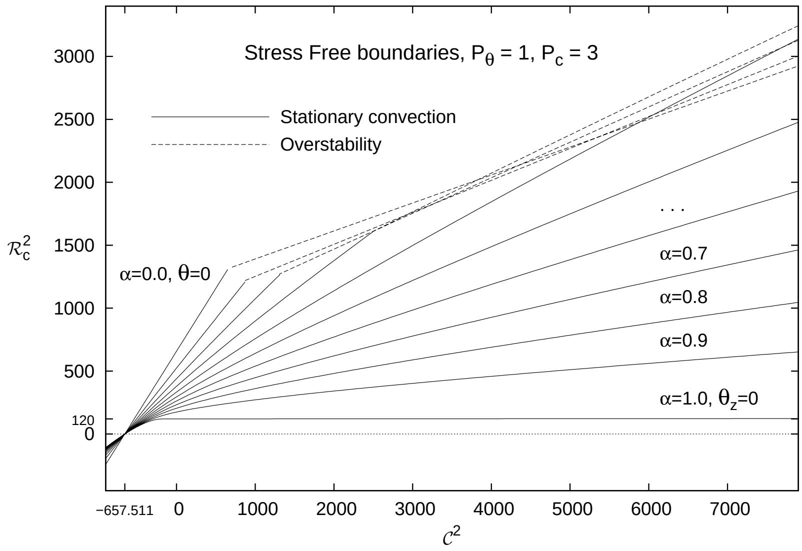


Figure 4.1: $\mathcal{R}_c^2$ as a function of $\mathcal{C}^2$ for thermal BCs going from fixed temperatures ($\alpha = 0$) to fixed heat fluxes ($\alpha = 1$), FF boundaries.

a stabilizing effect of rotation *even for fixed heat fluxes*). The result is nevertheless correct: an asymptotic analysis of the system for $a \to 0$ confirms that $\mathcal{R}_c^2 = 120$ (for FF boundaries), independently of $\mathcal{C}$. A point to note is that, since the critical Rayleigh number is independent of the concentration for fixed heat fluxes, we can use the classical non-linear energy stability analysis in the absence of a solute. The critical value $\mathcal{R}_c^2$ ensures then *global stability* (w.r.t. the classical energy norm) for any solute gradient. In Fig. 4.2 we show the critical wave number corresponding to the critical Rayleigh numbers of Fig. 4.1. For fixed heat fluxes, and for any stabilizing solute gradient ($\mathcal{C}^2 > 0$) the wave number is equal to zero. For fixed temperatures the critical wave number a_c is the constant $\pi/\sqrt{2}$. At the transition between stationary convection and overstability, for non-thermostatic boundaries, the wave number has a discontinuity. Fig. 4.3 shows the influence on stability of the boundary conditions on the solute. The results for fixed solute concentrations are analytically known (see e.g. Joseph (1976); Lombardo and Mulone (2002)), and are linear both in the convective and overstable regime.

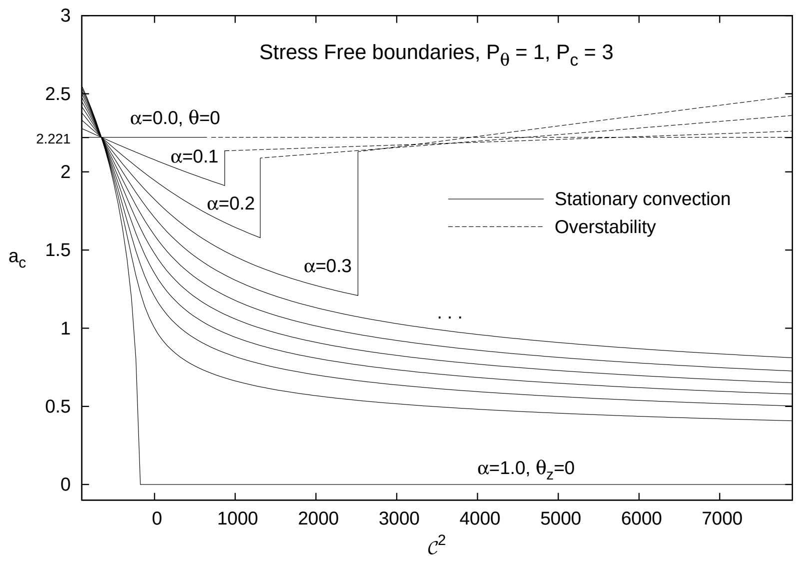


Figure 4.2: Critical wave number a_c as a function of $\mathcal{C}^2$ for thermal boundary conditions going from fixed temperatures ($\alpha = 0$) to fixed heat fluxes ($\alpha = 1$), FF boundaries. The vertical segments are added as a guide for the eye.

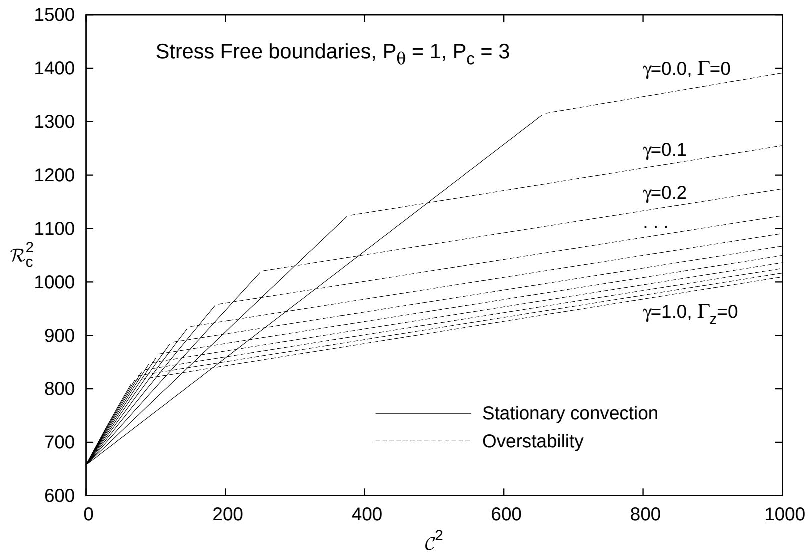


Figure 4.3: Critical Rayleigh number $\mathcal{R}_c^2$ as a function of $\mathcal{C}^2$ for fixed temperatures, solute boundary conditions going from fixed solute ($\gamma = 0$) to fixed solute fluxes ($\gamma = 1$) on both boundaries, hydrodynamic FF boundaries.

From the numerical results shown in the figure, the dependence on $\mathcal{C}^2$ seems linear also for Robin boundary conditions on the solute field, so we can suspect that an analytical solution exists even in this case. We observe also that in the convective regime the most stable condition is always obtained to fixed solute fields, while the situation is reversed in the case of overstability. The solute field remains in any case a stabilizing field. Contrary to the case of fixed heat fluxes, for fixed solute fluxes the critical Rayleigh number does not become independent on the solute gradient. Moreover, the corresponding critical wave number is not zero, so the long wavelength phenomenon seems linked only to a fixed flux of the *destabilizing* field.

4.4 Conclusions

The stability of a binary fluid layer subject to Neumann boundary conditions on the temperature, i.e. subject to fixed heat fluxes, results totally independent on the solute field. A clear physical interpretation of the phenomenon is yet to be found. The most stabilizing thermal boundary conditions, at least for stationary convection, are thermostatic boundaries. A better analysis of the transition between stationary convection and overstability is required, and a nonlinear stability analysis for the general Newton-Robin case is in progress.

Chapter 5

Solute and rotation

The stabilizing effect of uniform rotation has been predicted by Chandrasekhar (1961) in the *rotating* Bénard problem. A stabilizing effect is obtained even by salting the fluid layer from below (Joseph, 1976). Two or more simultaneously acting stabilizing effects allow observation of a very rich variety of phenomena often surprising. As is known, a *cooperative behaviour* has been showed among the rotation and salt concentration field (when the mixture is salted from below) (Lombardo, 2008). Here we study how the boundary conditions influence the interaction of rotation and salt (supplied from below), in particular Robin and Neumann boundary conditions on temperature. Numerical results are obtained with a Chebyshev-tau method (Dongarra et al., 1996).

5.1 The Bénard problem of a rotating mixture

Let $Oxyz$ be a cartesian frame of reference with unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ respectively, rotating at the constant velocity $\bar{\Omega}\mathbf{k}$. Let $d > 0$ and assume that a newtonian fluid is confined in the layer $\Omega_d = \mathbb{R}^2 \times (-d/2, d/2)$, and subject to a gravity field $\mathbf{g} = -g\mathbf{k}$. We assume also that the density of the fluid depends linearly on temperature T and concentration C of a solute according to $\rho_f = \rho_0[1 - \alpha_T(T - T_0) + \alpha_C(C - C_0)]$, with α_T, α_C positive coefficients of volume expansion and T_0, C_0 reference temperature and concentration (Chandrasekhar, 1961; Joseph, 1976; Mulone and Rionero, 1998). For the temperature field we assume Newton-Robin boundary conditions, which are linear combination of the temperature at a surface and its normal gradient. This boundary conditions describe the physical cases in which the media surrounding the fluid are not thermostatic (Falsaperla and Mulone, 2010; Joseph, 1976). The limit cases of fixed temperatures or fixed temperature gradients (and hence fixed heat fluxes) are also considered. We use the following general form of the thermal boundary conditions:

$$\begin{aligned} &\alpha_H(T_z + \beta_T)d + (1 - \alpha_H)(T_H - T) = 0, \qquad &\text{on } z = -d/2 \\ &\alpha_L(T_z + \beta_T)d + (1 - \alpha_L)(T - T_L) = 0, \qquad &\text{on } z = d/2, \end{aligned} \tag{5.1}$$

where $\alpha_H, \alpha_L \in [0, 1]$, $\beta_T > 0$, and $T_H = T_0 + \beta_T d/2$, $T_L = T_0 - \beta_T d/2$ are respectively an higher (T_H) and lower (T_L) temperature. Note that, from (5.1), we obtain fixed temperature, fixed heat flux, or a Newton-Robin boundary condition (Chapman and Proctor, 1980; Nield, 1964; Sparrow et al., 1963) at $z = d/2$, when α_L is equal to $0, 1$ or $0 < \alpha_L < 1$, respectively. The same observations apply to α_H and the boundary $z = -d/2$. For the velocity field, we assume that the boundaries are either rigid ($\mathbf{v} = 0$) or stress free ($\mathbf{k} \cdot \mathbf{v} = \partial_z(\mathbf{i} \cdot \mathbf{v}) = \partial_z(\mathbf{j} \cdot \mathbf{v}) = 0$) (Chandrasekhar, 1961). Concentrations at the boundaries are $C(x, y, -d/2) = C_0 + \beta_C d/2$, $C(x, y, d/2) = C_0 - \beta_C d/2$, where β_C is an assigned concentration gradient. The form of (5.1) ensures that for any choice of α_H, α_L, (and rigid or stress-free boundaries) the basic solution m_0 is the same, simplifying further analysis

$$\mathbf{v} = 0, \quad T(x, y, z) = -\beta_T z + T_0, \quad C(x, y, z) = -\beta_C z + C_0.$$

The non-dimensional evolution equations of a perturbation to the basic motionless state m_0 are (Lombardo, 2008)

$$\begin{cases} \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p^* + (\mathcal{R}\vartheta - \mathcal{C}\gamma)\mathbf{k} + \Delta \mathbf{u} + \mathcal{T}\mathbf{u} \times \mathbf{k}, \\ \nabla \cdot \mathbf{u} = 0, \ P_T(\vartheta_t + \mathbf{u} \cdot \nabla \vartheta) = \mathcal{R} w + \Delta \vartheta, \\ P_C(\gamma_t + \mathbf{u} \cdot \nabla \gamma) = \mathcal{C} w + \Delta \gamma \end{cases}$$

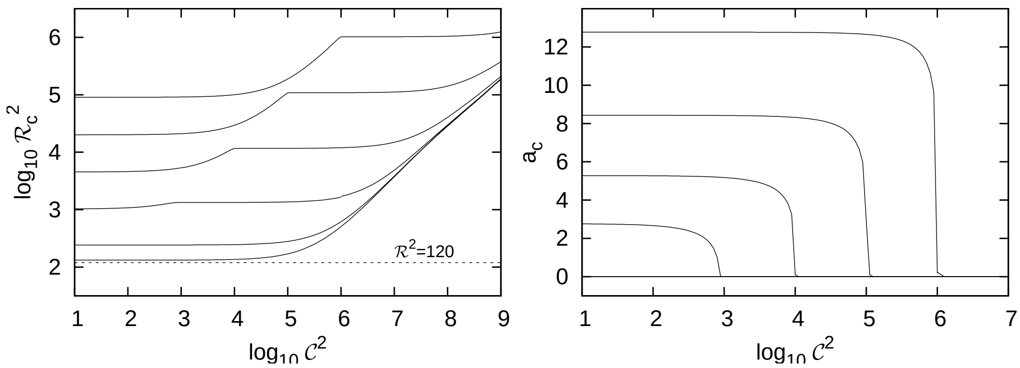


Figure 5.1: $\mathcal{R}_c^2$ and a_c as a function of $\mathcal{C}^2$ for fixed heat fluxes. Taylor number is equal to $10^6, 10^5, \ldots, 10$ from top to bottom in both graphics. For $\mathcal{T}^2 = 10, 100$, a_c is identically equal to zero for any $\mathcal{C}$.

in $\Omega_1 \times (0, \infty)$ where $\Omega_1 = \mathbb{R}^2 \times (-1/2, 1/2)$. In this system $\mathbf{u} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$, ϑ, γ and p^* are functions of (x, y, z, t) which represent the perturbations of the velocity, temperature, concentration and pressure fields, respectively; ∇ is the gradient operator and Δ is the Laplacian. The parameters $\mathcal{R}^2, \mathcal{C}^2, \mathcal{T}^2, P_T, P_C$ are the standard Rayleigh number for heat and solute, Taylor number, Prandtl and Schmidt numbers, respectively (Chandrasekhar, 1961; Joseph, 1976).

5.2 Instability equations

We study the linear instability of the basic motion, following Chandrasekhar (1961). We assume for the perturbation fields the general form, periodic in x, y, $f = F(z) \exp\{\, i\,(a_x x + a_y y) + p\,t\}$, where f denotes any of the fields $w, \zeta (= k \cdot \nabla \times \mathbf{u}), \theta, \gamma$ and $p = \sigma + i\tau$. Then, following standard calculationsChandrasekhar (1961); Lombardo (2008), and adopting a suitable rescaling of fields, we can derive the equations

$$\begin{cases} p\,(D^2 - a^2)W = (D^2 - a^2)^2 W - \mathcal{T} DZ + \mathcal{C} a^2\,\Gamma - \mathcal{R} a^2\,\Theta \\ pZ = \mathcal{T} DW + (D^2 - a^2)Z \\ p\,P_T\,\Theta = (D^2 - a^2)\Theta + \mathcal{R}\,W, \\ p\,P_C\,\Gamma = (D^2 - a^2)\Gamma + \mathcal{C}\,W, \end{cases} \tag{5.2}$$

where $a = (a_x^2 + a_y^2)^{1/2}$ is the wave number, and D^n denotes the n-th partial derivative respect to z. The boundary conditions for system (5.2) are $W = \Gamma = 0$ on $z = \pm 1/2$, $D^2 W = DZ = 0$ on stress-free boundaries, $DW = Z = 0$ on rigid boundaries, $\alpha_H D\Theta - (1 - \alpha_H)\Theta = 0$ on $z = -1/2$ and $\alpha_L D\Theta + (1 - \alpha_L)\Theta = 0$ on $z = 1/2$. When the Principle of Exchange of Stabilities (PES) holds ($\sigma = 0 \Rightarrow \tau = 0$), for stress-free and thermostatic boundaries, it is possible to find (Lombardo, 2008) for the critical Rayleigh number

$$\mathcal{R}_1^2 = \frac{(1+x)^3}{x} + \frac{\mathcal{T}_1^2}{x} + \mathcal{C}_1^2, \tag{5.3}$$

where $\mathcal{R}_1 = \mathcal{R}/\pi^2$, $x = a^2/\pi^2$, $\mathcal{C}_1 = \mathcal{C}/\pi^2$, and $\mathcal{T}_1 = \mathcal{T}/\pi^2$. In (5.3), the second and third term are exactly the stabilizing contributes appearing when only one of the two fields is present (Chandrasekhar, 1961; Joseph, 1976), moreover the critical wave number appears independent of $\mathcal{C}$.

5.3 Results and conclusions

We consider the case of fixed heat fluxes, stress-free boundaries and $P_T = P_C = 1$. At a difference from the case $\mathcal{T} = 0$ (Falsaperla and Mulone, 2009), where $\mathcal{R}^2$ is constant ($\mathcal{R}^2 = 120$, dashed line in Fig. 1a), now the Rayleigh number is an increasing function of $\mathcal{C}$ and $\mathcal{T}$, and so the stabilizing effect of the solute is restored. Angular points correspond to a transition to a region of vanishing values of a_c. Moreover, we observe (Fig. 1b) a "competition" on the wave number between rotation and concentration gradient, in the sense that the wave number, for any fixed $\mathcal{T}$, becomes zero for sufficiently large values of $\mathcal{C}$; on the other hand, for fixed $\mathcal{C}$ and sufficiently large $\mathcal{T}$, it is $a_c > 0$.

The same figure shows that, for large values of $\mathcal{T}, \mathcal{C}$, the region $a_c = 0$ is very closely defined by $\mathcal{C}^2 > \mathcal{T}^2$.

It is possible to check that the concentration and rotation fields remain cooperative in their stabilizing effects, for any $\alpha_H, \alpha_L \in [0, 1]$, and different values of P_T, P_C. For $P_C > P_T$ overstability effects appear. At least for stationary convection, thermal boundary conditions are more destabilizing as the parameters α_H, α_L increase, i.e. in the transition from fixed temperatures to fixed heat fluxes.

Chapter 6

Magnetic fluids

A stabilizing effect for the Bénard problem is obtained immersing it in a normal magnetic field, if the fluid is electrically conducting, Chandrasekhar (1961). As is known, unexpected *conflicting tendencies* among rotation and magnetic field have been found by Chandrasekhar. Here we study the interaction of magnetic field and rotation, coupled with Robin and Neumann boundary conditions on temperature.

6.1 The rotating magnetic Bénard problem

The magnetic Bénard problem deals with the onset of convection in a horizontal layer of a homogeneous, viscous, and electrically conducting fluid, permeated by an imposed uniform magnetic field normal to the layer, and heated from below (Chandrasekhar, 1961; Thompson, 1951).

We suppose here that the system has the same geometry considered in the previous section, and the corresponding fields are subject to the same boundary conditions. Following the procedure of Chandrasekhar (1961), Chapter V, we arrive to the linear instability analysis system: where X and K are

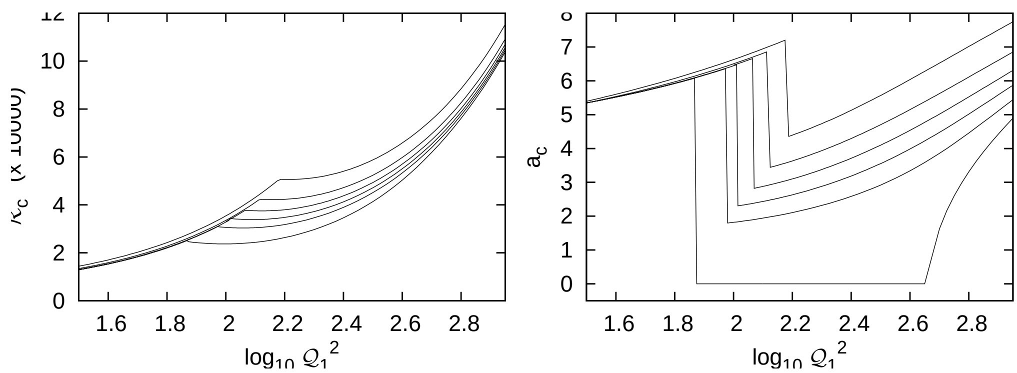


Figure 6.1: $\mathcal{R}_c^2$ and a_c as a function of $\mathcal{Q}_1^2$ for $\mathcal{T}_1^2 = 10^4$. Thermal boundary conditions vary in both graphics from fixed temperatures (top curves) to fixed heat fluxes (bottom curves), with $\alpha_H = \alpha_L = 0, 0.2, 0.4, 0.6, 0.8, 1$. The vertical segments in the a_c graphs correspond to transitions from overstability to stationary convection. The same transition appears as a discontinuity in the slope of the $\mathcal{R}_c^2$ graphs.

the perturbations

$$\begin{cases} (D^2 - a^2)(D^2 - a^2 - p)W + D(D^2 - a^2)K - DZ - a^2\,\Theta = 0, \\ (D^2 - a^2 - p)Z + \mathcal{T}^2 DW + DX = 0, \\ (D^2 - a^2 - P_m p)X + \mathcal{Q}^2\,DZ = 0, \\ (D^2 - a^2 - P_m p)K + \mathcal{Q}^2\,DW = 0, \\ (D^2 - a^2 - P_T p)\Theta + \mathcal{R}^2\,W = 0, \end{cases} \tag{6.1}$$

to the third component of current density and H (the imposed magnetic field), $\mathcal{Q}^2$ and P_m are the Chandrasekhar and magnetic Prandtl numbers (Chandrasekhar, 1961). When PES holds, it is possible to set $p = 0$ and eliminate field K from (6.1). The same elimination is possible for $P_m = 0$ (this happens to a good approximation (Chandrasekhar, 1961) for liquid metals, e.g. mercury), and our stability analysis is performed under this hypothesis. On X, we impose $DX = 0$ or $X = 0$ for electrically conducting or non-conducting boundaries, respectively (Chandrasekhar, 1961).

In the analytically solvable case of stress-free, thermostatic and non-conducting boundaries, Chandrasekhar finds (see Chandrasekhar (1961), Chap.

V eq. 59) for stationary convection

$$\mathcal{R}_1^2 = \frac{(1+x)^3}{x} + \frac{\mathcal{T}_1^2}{x} + \frac{(1+x)\mathcal{Q}_1^2}{x} - \frac{\mathcal{Q}_1^2\mathcal{T}_1^2}{x((1+x)^2+\mathcal{Q}_1^2)},$$

where $\mathcal{Q}_1 = \mathcal{Q}/\pi$. The second and third terms in the previous expression can be found when the system is subject only to rotation or a magnetic field (see Chandrasekhar (1961), Chap. III eq. 130, Chap. IV eq. 165). The presence of the last term shows that the two effects are not simply additive.

6.2 Results and conclusions

For stress free boundaries and $P_T = 0.025$ (mercury), competition of magnetic field and rotation is enhanced by the new thermal boundary conditions, and appears clearly in Fig. 2a where the slope becomes negative. We can observe also in Fig. 2b a dramatic reduction of the critical parameter a_c when heat flux is prevalent. We note that $a_c = 0$ in a finite range of values of $\mathcal{Q}$ for Neumann conditions on temperature.

A more extensive study of the system, for different values of the Prandtl numbers and other hydrodynamic and magnetic boundary conditions will be the subject of a future work.

Chapter 7

Analytic approach to the solute Bénard system

The solute Bénard problem (introduced in the previous chapters) depends on many physical parameters, which are related to properties of the fluid, and to characteristics of the boundaries. Here, in particular, we assume very general boundary conditions: finite–slip conditions on the velocity (Baudry and Charlaix, 2001; Craig et al., 2001; Navier, 1823; Webber, 2007), Newton–Robin conditions on the temperature (Busse and Riahi, 1980; Chapman and Proctor, 1980; Clever and Busse, 1998; Hurle et al., 1966; Nield, 1964) and Robin conditions on the solute concentration (Joseph, 1976).

This problem admits a motionless solution in which temperature and concentration have constant vertical gradient. A typical method to discuss the stability of a solution, is to investigate its linear instability by computing the equations of its perturbations. Such equations reduce, after some manipulations, under the standard assumption of periodicity in the x, y directions, and assuming the validity of the principle of exchange of stabilities, to a linear system depending on three essential non–dimensional parameters: the wave number a, the Rayleigh number R and the solute Rayleigh number C. The problem also depends on additional parameters appearing in the boundary conditions.

Our goal is to write analytic equations to describe the region in parameter space where a change of stability can possibly happen; such region is called *marginal region.* More precisely, we show that the marginal region is the zero level–set of analytic functions, that we call *marginal functions.* Such functions depend on the parameters (a, R, C) via algebraic functions (polynomial and radicals) composed with circular and hyperbolic functions, and depend on boundary parameters linearly. Considering these last parameters as accessory, we investigate marginal functions for particular choices which correspond to extremal cases: *rigid, stress-free, fixed temperature, fixed heat flux, fixed solute concentration, and fixed flux of solute.* We also deduce the asymptotic behavior of the marginal region for vanishing wave number; in particular, we show that the marginal region approaches the axes $a = 0$ only for fixed temperature gradient at both boundaries (see Falsaperla and Giacobbe, 2010).

7.1 Equations and boundary conditions

In a cartesian frame of reference $Oxyz$ with unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, consider a layer $\Omega_d = \mathbb{R}^2_{x,y} \times (-d/2, d/2)_z$ of thickness $d > 0$, filled with a newtonian fluid, and subject to the gravity field $\mathbf{g} = -g\mathbf{k}$. In the Oberbeck-Boussinesq approximation, the equations governing the motion of the fluid are given by (see Joseph, 1976)

$$\begin{cases} (u, v, w)_t + (u, v, w) \cdot \nabla(u, v, w) + \dfrac{\nabla p}{\rho_0} - \dfrac{\rho}{\rho_0}\mathbf{g} = \nu\Delta(u, v, w) \\ \nabla \cdot (u, v, w) = 0 \\ \vartheta_t + (u, v, w) \cdot \nabla\vartheta = \kappa_\vartheta \Delta\vartheta \\ \gamma_t + (u, v, w) \cdot \nabla\gamma = \kappa_\gamma \Delta\gamma \end{cases}$$

where (u, v, w) is the velocity, ϑ is the temperature, γ is the solute concentration, and p is the pressure, while ν, κ_ϑ, and κ_γ are positive constants which represent kinematic viscosity, and thermal and solute diffusivity. The equations are written making the assumption that the fluid density ρ depends on temperature and solute concentration according to the linear law

$$\rho(\vartheta,\gamma)=\rho_0[1-c_\vartheta(\vartheta-\vartheta_0)+c_\gamma(\gamma-\gamma_0)],$$

obtained by a Taylor expansion of the density function about a reference temperature ϑ_0 and concentration γ_0. Here ρ_0 is the reference density, and c_ϑ, c_γ are positive density variation coefficients for temperature and concentration respectively. Gradient, Laplacian and derivative with respect to time are represented respectively by the symbols ∇, Δ, and the subscript "$_t$".

Other than the *containment condition* $w=0$ on both boundaries, a reasonably simple and yet very general set of boundary conditions for the velocity field is the so called *finite–slip condition* (Baudry and Charlaix, 2001; Craig et al., 2001), which are

$$\begin{cases} u-\lambda_-(\partial_x w+\partial_z u)=0, \quad v-\lambda_-(\partial_y w+\partial_z v)=0, & \text{on } z=-d/2,\\ u+\lambda_+(\partial_x w+\partial_z u)=0, \quad v+\lambda_+(\partial_y w+\partial_z v)=0, & \text{on } z=d/2. \end{cases}$$

The limit condition $\lambda_\pm=0$ is the *rigid boundary condition*, and imposes zero tangential velocity at the corresponding boundary. The limit $\lambda_\pm=\infty$ is the *stress free condition*, and imposes vanishing tangential stresses at the corresponding boundary. For temperature, a general set of boundary conditions are the so called *Newton–Robin boundary conditions* (Hurle et al., 1966; Nield, 1964), which describe the physically relevant cases in which the media surrounding the fluid is not an ideal thermostat (see also Joseph (1976)). In the literature, many explicit forms of the Newton–Robin boundary conditions are used, but we find convenient to choose them in such a way that, by varying a single coefficient, different thermal boundary conditions can be obtained, but the basic motionless solution is preserved:

$$\begin{cases} \alpha_-(\vartheta_z+G_\vartheta)-(\vartheta-\vartheta_\pm)=0, & \text{on } z=-d/2,\\ \alpha_+(\vartheta_z+G_\vartheta)+(\vartheta-\vartheta_\pm)=0, & \text{on } z=d/2. \end{cases}$$

Here, $\alpha_\pm$ and G_ϑ are non–negative real numbers, $\vartheta_-=\vartheta_0+G_\vartheta d/2$, $\vartheta_+=\vartheta_0-G_\vartheta d/2$ are respectively the temperatures at the lower (ϑ_-) and higher (ϑ_+)

boundaries and ϑ_0 is the reference temperature (observe that the boundary conditions depend on $\alpha_\pm$ and on two other parameters not on three, as it may seem). The limit $\alpha_\pm = 0$ is the *infinite conductivity* boundary condition, in which is fixed the value of the temperature at the corresponding boundary, while the limit $\alpha_\pm = \infty$ is the *insulating* boundary condition (Busse and Riahi, 1980; Chapman and Proctor, 1980; Clever and Busse, 1998), with a fixed heat flux $\mathbf{q}$ of norm $q = G_\vartheta \kappa_\vartheta$ and directed along the z-axis at the corresponding boundary. The cases in which $\alpha_\pm$ is finite and positive are cases of finite conductivity at the corresponding boundary.

For the solute field, by similar considerations (Joseph, 1976), we use boundary conditions depending on both concentration of solute and its normal derivative at the boundary surfaces. Again, to obtain a range of boundary conditions (from fixed concentrations to fixed fluxes of solute) depending on a single parameter on each boundary, while maintaining the basic solution, we use the expressions

$$\begin{cases} \beta_-(\gamma_z + G_\gamma) - (\gamma - \gamma_\pm) = 0, & \text{on } z = -d/2, \\ \beta_+(\gamma_z + G_\gamma) + (\gamma - \gamma_\pm) = 0, & \text{on } z = d/2, \end{cases}$$

where $\beta_\pm, G_\gamma$ are non–negative real numbers, $\gamma_+ = \gamma_0 - G_\gamma d/2$, $\gamma_- = \gamma_0 + G_\gamma d/2$ are respectively the concentration of solute at the lower (γ_-) and upper (γ_+) boundary, and γ_0 is the reference density.

The steady solution whose stability we wish to investigate is, for any choice of the parameters $\alpha_\pm, \beta_\pm$, the state $(\bar{u}, \bar{v}, \bar{w}, \bar{\vartheta}, \bar{\gamma}, \bar{p})$ with

$$\begin{gathered} \bar{u} = \bar{v} = \bar{w} = 0, \quad \overline{\vartheta} = \vartheta_0 - G_\vartheta z, \quad \overline{\gamma} = \gamma_0 - G_\gamma z, \\ \bar{p} = p_0 - \rho_0 g z - \frac{\rho_0 g}{2}(c_\vartheta G_\vartheta - c_\gamma G_\gamma) z^2. \end{gathered} \tag{7.1}$$

Note that in (4.3) G_ϑ, G_γ are the temperature and concentration gradients.

The equations which govern the evolution of a disturbance to (4.3) are

$$\begin{cases} (U,V,W)_t + (U,V,W)\cdot\nabla(U,V,W) + \dfrac{\nabla\Pi}{\rho_0} + \\ \qquad\qquad\qquad + (c_\vartheta\Theta - c_\gamma\Gamma)\mathbf{g} = \nu\Delta(U,V,W), \\ \nabla\cdot(U,V,W) = 0, \\ \Theta_t + (U,V,W)\cdot\nabla\Theta = G_\vartheta W + \kappa_\vartheta\Delta\Theta, \\ \Gamma_t + (U,V,W)\cdot\nabla\Gamma = G_\gamma W + \kappa_\gamma\Delta\Gamma, \end{cases}$$

where (U, V, W), Θ, Γ, Π are the perturbations to the velocity, temperature, concentration, and pressure fields, respectively.

Following the standard linear instability analysis of Chandrasekhar (1961), applying twice the curl operator to the first equation, and then considering only the linear terms of the resulting systems, one obtains

$$\begin{cases} \Delta U_t = -gc_\vartheta\Theta_{xz} + gc_\gamma\Gamma_{xz} + \nu\Delta^2 U, \\ \Delta V_t = -gc_\vartheta\Theta_{yz} + gc_\gamma\Gamma_{yz} + \nu\Delta^2 V, \\ \Delta W_t = g\Delta^*(c_\vartheta\Theta - c_\gamma\Gamma) + \nu\Delta^2 W, \\ \Theta_t = G_\vartheta W + \kappa_\vartheta\Delta\Theta, \\ \Gamma_t = G_\gamma W + \kappa_\gamma\Delta\Gamma, \end{cases} \tag{7.2}$$

where $\Delta^\star = \partial^2/\partial x^2 + \partial^2/\partial y^2$. The last three equations are autonomous, and the system is solved by determining the solution fields W, Θ, Γ and then substituting them in the first two equations, that can always be solved (Chandrasekhar, 1961, II §10). The boundary condition for the three functions W, Θ, Γ are

$$\begin{cases} W = 0, \quad \partial_z W - \lambda_-\partial_z^2 W = 0, \quad \Theta - \alpha_-\partial_z\Theta = 0, \quad \Gamma - \beta_-\partial_z\Gamma = 0, \\ W = 0, \quad \partial_z W + \lambda_+\partial_z^2 W = 0, \quad \Theta + \alpha_+\partial_z\Theta = 0, \quad \Gamma + \beta_+\partial_z\Gamma = 0, \end{cases} \tag{7.3}$$

respectively on $z = \pm d/2$. As a last step assume, as usual, that the perturbation fields are sufficiently smooth, and that they are periodic in the x and y directions. Substituting the functions W, Θ, Γ in (7.2) with functions of the form

$$W(z)e^{i\,(k_x x + k_y y)}e^{(\sigma+i\tau)\,t},\ \Theta(z)e^{i\,(k_x x + k_y y)}e^{(\sigma+i\tau)\,t},\ \Gamma(z)e^{i\,(k_x x + k_y y)}e^{(\sigma+i\tau)\,t}$$

(with some abuse of notation), and denoting by $k = (k_x^2 + k_y^2)^{1/2}$ the wave number, one obtains the following system of ODE for the perturbation fields W, Θ, Γ, which are now functions of z only and where D is the derivation with respect to z

$$\begin{cases} (\sigma + i\tau)\,(D^2 - k^2)W = -gk^2(c_\vartheta\Theta - c_\gamma\Gamma) + \nu(D^2 - k^2)^2 W, \\ (\sigma + i\tau)\,\Theta = G_\vartheta W + \kappa_\vartheta(D^2 - k^2)\Theta, \\ (\sigma + i\tau)\,\Gamma = G_\gamma W + \kappa_\gamma(D^2 - k^2)\Gamma. \end{cases} \tag{7.4}$$

A change in the stability of the purely conducting solution can take place only for a choice of parameters for which there exists a solution to the equations (7.4), (7.3) in which the real part σ of the eigenvalue is equal to zero. It is hence possible to explicitly define the *marginal region* as the subset of parameter space in which there exist non–zero solutions of equations (7.4), (7.3) with $\sigma = 0$. Naturally, the points that separate linear stability from instability are a subset of the marginal region. Here, we determine marginal states which satisfy the further assumption that the principle of exchange of stabilities holds, that is, marginal states at which both $\sigma = 0$ and $\tau = 0$.

By posing $dz_1 = z$, $a = kd$, $W_1 = \nu W/d^2$, $\Theta_1 = gc_\vartheta\Theta$, $\Gamma_1 = gc_\gamma\Gamma$, introducing $R^2 = gc_\vartheta G_\vartheta d^4/(\nu\kappa_\vartheta)$, $C^2 = gc_\gamma G_\gamma d^4/(\nu\kappa_\gamma)$ (the Rayleigh number and the solute Rayleigh number respectively), and eliminating the subscript from $z_1, W_1, \Theta_1, \Gamma_1$, equations (7.4) can be cast in the non–dimensional form given in §56 of Joseph (1976),

$$\begin{cases} 0 = -a^2\Theta + a^2\Gamma + (D^2 - a^2)^2 W, \\ 0 = R^2 W + (D^2 - a^2)\Theta, \\ 0 = C^2 W + (D^2 - a^2)\Gamma. \end{cases} \tag{7.5}$$

The set of points (a, R, C) of the parameter space for which there exist non–zero solutions to equations (7.5) satisfying boundary conditions

$$W = 0, \quad DW \pm \lambda_\pm D^2 W = 0, \quad \Theta \pm \alpha_\pm D\Theta = 0, \quad \Gamma \pm \beta_\pm D\Gamma = 0 \tag{7.6}$$

on $z = \pm 1/2$ (with λ, α, β substituted by their non–dimensional version), is the object of our investigation.

The original content of the following results consists in calculating analytic functions (the marginal functions) whose zero level–set is the marginal region and in using the equations to obtain information on such region. We postpone computation and investigation of the marginal functions to Section 7.3, while we devote Section 7.2 to the technique that allows us to compute such functions.

7.2 A general approach to linear boundary value problems

A real *linear boundary value problem* is the system of equations

$$V'(z) = AV(z), \tag{7.7a}$$

$$V(-1/2) \in \mathcal{C}_-, \quad V(1/2) \in \mathcal{C}_+ \tag{7.7b}$$

where $V(z) : [-1/2, 1/2] \to \mathbb{R}^n$ is a vector–valued function, A is a real $n \times n$ matrix, $\mathcal{C}_-, \mathcal{C}_+$ are two subspaces of $\mathbb{R}^n$ of dimension n_-, n_+ respectively. Our goal is to give conditions on the matrix A and the subspaces $\mathcal{C}_\pm$ under which non–zero solutions to (7.7) exist. We will show that the existence of a non–zero solution is granted by the vanishing of analytic expressions depending on the matrix A and the vector spaces $\mathcal{C}_\pm$.

Remark 7.1. *In the problem we wish to investigate, the matrix A depends on a, R, C, the vector space $\mathcal{C}_-$ depends on $\lambda_-, \alpha_-, \beta_-$, and the vector space $\mathcal{C}_+$ depends on $\lambda_+, \alpha_+, \beta_+$.*

7.2.1 The general case

The general solution to the linear differential equation (7.7a) is of the form $e^{zA}v$, where $v \in \mathbb{R}^n$. It is hence straightforward that a non–zero solution to (7.7a) also satisfies (7.7b) if and only if $e^A\mathcal{C}_- \cap \mathcal{C}_+ \neq \{0\}$.

The fact that $e^A\mathcal{C}_-$ and $\mathcal{C}_+$ have at least a one–dimensional intersection implies that, letting B be a $n \times n_-$ matrix whose columns form a basis of $\mathcal{C}_-$, and letting P be a $(n - n_+) \times n$ matrix whose rows are a basis of $\mathcal{C}_+^\perp$, there exists a non–zero $w \in \mathbb{R}^{n_-}$ such that $Pe^A Bw = 0$.

Assume now that $n = 2m$, and that $n_- = n_+ = m$. Then the matrix $Pe^A B$ is a $m \times m$ matrix, and the existence of a non–zero w in its kernel is equivalent to the determinant being zero. Hence,

$$\det(Pe^A B) = 0 \tag{7.8}$$

is the *existence condition* of non–zero solutions to (7.7).

Remark 7.2. *In our case, the vector spaces $\mathcal{C}_\pm$ depend linearly on the respective parameters $\lambda_\pm, \alpha_\pm, \beta_\pm$ (each with the corresponding sign), and such parameters appear in only one entry of the matrices P and B. It follows that* $\det(Pe^A B)$ *is a function that depends linearly on $\lambda_\pm, \alpha_\pm, \beta_\pm$. The dependence of* $\det(Pe^A B)$ *on (a, R, C) is due to e^A, and is a composition of algebraic functions with circular or hyperbolic functions.*

7.2.2 The parity preserving case

There is a yet more sophisticated method to adopt, that yields much simpler existence conditions. In the hypothesis that $n = 2m$, let J be the diagonal matrix whose entries in the diagonal are an alternating sequence of $+1$ and -1, i.e. $J = \mathrm{diag}(1, -1, \dots, 1, -1)$. We then say that

Definition 7.3. *The boundary value problem is* parity–preserving *if $JAJ = -A$ and $J\mathcal{C}_- = \mathcal{C}_+$.*

The matrix JAJ has entries in position i, j equal to those of A when $i - j$ is even, and entries in position i, j opposite to those of A when $i - j$ is odd. It follows that $JAJ = -A$ if and only if A has non–zero entries only in the positions i, j such that $i - j$ is odd. Parity–preserving linear boundary value problems are typical: every differential equations which contains only even derivatives has associated system of differential equations which is parity–preserving. It is immediate to prove the following fact.

Proposition 7.4. *Let $V(z)$ be a solution of a parity–preserving linear boundary value problem. Then the functions $V_e(z) = (V(z) + JV(-z))/2$, $V_o(z) = (V(z) - JV(-z))/2$ are also solutions of the linear boundary value problem. We call such expressions respectively the even and odd part of $V(z)$.*

Proof. Assume that the vector–valued function $V(z)$ is a solution of equation (7.7a). Then, necessarily, $V(z) = e^{zA}v$, and hence $V_e(z) = (e^{zA} + Je^{-zA})v/2$. Computing the z-derivative of $V_e(z)$ one obtains that

$$V_e'(z) = \frac{1}{2}(Ae^{zA} - JAe^{-zA})v = \frac{1}{2}(Ae^{zA} + AJe^{-zA})v = AV_e(z).$$

If moreover $V(z)$ satisfies the boundary conditions (7.7b), that is $V(-1/2) \in \mathcal{C}_-$ and $V(1/2) \in \mathcal{C}_+$, it follows that

$$V_e\left(-\frac{1}{2}\right) = \frac{1}{2}\left(V\left(-\frac{1}{2}\right) + JV\left(\frac{1}{2}\right)\right) \in \mathcal{C}_- + J\mathcal{C}_+ = \mathcal{C}_-,$$
$$V_e\left(\frac{1}{2}\right) = \frac{1}{2}\left(V\left(\frac{1}{2}\right) + JV\left(-\frac{1}{2}\right)\right) \in \mathcal{C}_+ + J\mathcal{C}_- = \mathcal{C}_+.$$

This, with an identical argument for the odd part, concludes the proof. □

Given a vector valued function $V(z)$, the vector valued function $V_e(z)$ is the one whose even entries are the odd part of the corresponding entries of $V(z)$ and whose odd entries are the even part of the corresponding entries of $V(z)$ (for $V_o(z)$ the same statement is true with the words even and odd exchanged). This observation is commonly used when dealing with systems of differential equations which contain only even derivatives (e.g. in Chandrasekhar (1961)). In such a situation, the space of solutions is the direct sum of even and odd solutions.

From the proposition above, we can conclude that all even (respectively odd) solutions of a parity–preserving boundary value problem are linear combinations of the columns of the matrices $e^{zA} + Je^{-zA}$ (respectively $e^{zA} - Je^{-zA}$), that we call *even generators* (respectively *odd generators*). A simple algebraic computation proves that the even generators are simply the odd columns of e^{zA}, while the odd generators are the even columns of e^{zA}. They both generate a m–dimensional space.

Let us denote by $E_1, \ldots, E_m$ the even generators and by $O_1, \ldots, O_m$ the odd generators. As in the previous section, there exists a non–zero even solution $w^i E_i(z)$ (with implicit summation over the index i) to the linear boundary value problem if and only if there exists a non–zero m–dimensional vector w such that $w^i E_i(1/2) \in \mathcal{C}_+$. This is equivalent to the fact that, denoting by E the $n\times m$ matrix whose columns are the vectors $E_i(1/2)$, and denoting by P the $m\times n$ matrix whose rows are a basis of $\mathcal{C}_+^\perp$, there must exist a w such that $PEw = 0$. Taking into account also the odd counterpart, the existence of a non–zero w in the kernel, is equivalent to the scalar conditions

$$\det(PE) = 0, \quad \det(PO) = 0, \tag{7.9}$$

which are the refined version of the existence conditions found in the previous section. These two equations are typically much simpler functions of the parameters.

Remark 7.5. *Observe that to compute equations* (7.9) *one needs not compute the matrix* e^{zA} *(which satisfies* $e^0 = \mathbb{I}_n$*), but only determine* m *independent even (respectively odd) generators of the space of solutions. On the other hand, computation of* e^{zA} *typically requires the pre and post composition with the* A*-diagonalizing matrix and its inverse, which is not necessary with this approach.*

7.3 Applications to the solute Bénard problem

Let us finally apply the arguments above to the solute Bénard problem. The general case produces formulas too long to be written, for this reason, we devote Subsection 7.3.1 to the parity–preserving case, which requires that the boundaries have the same characteristic constants ($\lambda_+ = \lambda_- = \lambda$, $\alpha_+ = \alpha_- = \alpha$, $\beta_+ = \beta_- = \beta$). Once computed the equations for the general marginal region, we draw them in some particular cases. In Subsection 7.3.2, we use the general approach to discuss the asymptotic behavior of the marginal region as a tends to zero.

At the end of Section 7.1, we deduced that a point in the effective parameter space $\mathbb{R}_a^+ \times \mathbb{R}_R^+ \times \mathbb{R}_C^+$ is a marginal state (under the assumption that the principle of exchange of stabilities holds) when and only when there exist non–zero solutions of the linear system of equations (7.5) with boundary conditions (7.6).

Denoting $V = (W, DW, D^2W, D^3W, \Theta, D\Theta, \Gamma, D\Gamma)$, the 8×8 matrix associated to the linear system of equations (7.5) is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -a^4 & 0 & 2a^2 & 0 & a^2 & 0 & -a^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -R^2 & 0 & 0 & 0 & a^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -C^2 & 0 & 0 & 0 & 0 & 0 & a^2 & 0 \end{bmatrix},$$

and its characteristic polynomial is

$$\left(\lambda^2 - a^2\right)\left(\left(\lambda^2 - a^2\right)^3 + a^2\left(R^2 - C^2\right)\right).$$

Denoting by b the only real cubic root of $a^2(R^2 - C^2)$, the eigenvalues of the matrix A are square roots of

$$a^2, \quad a^2 - b, \quad \left(a^2 + \frac{b}{2}\right) \pm ib\frac{\sqrt{3}}{2}.$$

It is hence clear that, excluding the degenerate cases $b = 0$, i.e. $R = C$ (in which case the characteristic polynomial has two quadruple roots equal to $\pm a$ and the Jordan form of A has two rank four Jordan blocks), and $b = a^2$ (in which case 0 is a double eigenvalue and the canonical form of A has a rank two Jordan block), the matrix A has always two real eigenvalues $\pm a$, four complex eigenvalues $\pm e_p \pm ie_m$, where

$$e_p = \frac{\sqrt{2d+b+2a^2}}{2}, \quad e_m = \frac{\sqrt{2d-b-2a^2}}{2}, \quad d = \sqrt{a^4+a^2b+b^2},$$

and two eigenvalues that are real if $a^2 - b > 0$ or pure imaginary if $a^2 - b < 0$. Let us denote $c = \sqrt{a^2 - b}$, with the understanding that c is either positive real or purely imaginary with positive imaginary part.

7.3.1 Applications of the parity–preserving formula: marginal equations and marginal regions

The matrix A is obviously parity preserving, assuming $\lambda_- = \lambda_+ = \lambda$, $\alpha_- = \alpha_+ = \alpha$ and $\beta_- = \beta_+ = \beta$, also the parity requirement on the boundary conditions is satisfied, and the system is parity–preserving. The matrix P introduced in Subsection 7.2.1 is

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \beta \end{bmatrix}$$

while, a basis of the even solutions introduced in Subsection 7.2.2 are

$$E_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \cosh(az) \\ a\sinh(az) \\ \cosh(az) \\ a\sinh(az) \end{bmatrix}, \quad E_2 = \begin{bmatrix} b\cosh(cz) \\ bc\sinh(cz) \\ bc^2\cosh(cz) \\ bc^3\sinh(cz) \\ R^2\cosh(cz) \\ cR^2\sinh(cz) \\ C^2\cosh(cz) \\ cC^2\sinh(cz) \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 2b\cos(ze_m)\cosh(ze_p) \\ 2b(e_p\cos(ze_m)\sinh(ze_p) - e_m\sin(ze_m)\cosh(ze_p)) \\ b\Big(\big(2a^2+b\big)\cos(ze_m)\cosh(ze_p) - \sqrt{3}b\sin(ze_m)\sinh(ze_p)\Big) \\ -2b\Big(e_p\big(-2a^2-b+d\big)\cos(ze_m)\sinh(ze_p) + e_m\big(2a^2+b+d\big)\sin(ze_m)\cosh(ze_p)\Big) \\ -R^2\big(\sqrt{3}\sin(ze_m)\sinh(ze_p) + \cos(ze_m)\cosh(ze_p)\big) \\ 0 \\ -C^2\big(\sqrt{3}\sin(ze_m)\sinh(ze_p) + \cos(ze_m)\cosh(ze_p)\big) \\ 0 \end{bmatrix},$$

$$E_4 = \begin{bmatrix} 2b\sin(ze_m)\sinh(ze_p) \\ 2b(e_m\cos(ze_m)\sinh(ze_p)+e_p\sin(ze_m)\cosh(ze_p)) \\ b\Big(\big(2a^2+b\big)\sin(ze_m)\sinh(ze_p)+\sqrt{3}b\cos(ze_m)\cosh(ze_p)\Big) \\ 2b\Big(e_m\big(2a^2+b+d\big)\cos(ze_m)\sinh(ze_p)+e_p\big(2a^2+b-d\big)\sin(ze_m)\cosh(ze_p)\Big) \\ R^2\big(\sqrt{3}\cos(ze_m)\cosh(ze_p)-\sin(ze_m)\sinh(ze_p)\big) \\ 0 \\ C^2\big(\sqrt{3}\cos(ze_m)\cosh(ze_p)-\sin(ze_m)\sinh(ze_p)\big) \\ 0 \end{bmatrix}.$$

The vectors $O_1, ..., O_4$ have very similar expressions, which we do not write here. A computation carried over with computer assisted algebra [1] yields a marginal function that is, in the even case, the function

$$C^2 p_\alpha(a) q_{\beta,\lambda}(a, R^2 - C^2) - R^2 p_\beta(a) q_{\alpha,\lambda}(a, R^2 - C^2), \tag{7.10}$$

where

$$p_\alpha = \cosh\frac{a}{2} + \alpha\, a \sinh\frac{a}{2}$$

$$\begin{aligned} q_{\alpha,\lambda} = \cosh\frac{c}{2}\Big[&(3b\lambda + d\alpha)\cosh e_p + (3b\lambda - d\alpha)\cos e_m + \\ &+ (\sqrt{3}e_m + (2b\alpha\lambda + 1)e_p)\sinh e_p + (\sqrt{3}e_p - (2b\alpha\lambda + 1)e_m)\sin e_m\Big] + \\ &+ c\sinh\frac{c}{2}\Big[(b\alpha\lambda - 1)\cos e_m + (b\alpha\lambda - 1)\cosh e_p + \\ &+ \alpha(\sqrt{3}e_m - e_p)\sinh e_p + \alpha(\sqrt{3}e_p + e_m)\sin e_m\Big] \end{aligned}$$

In the odd case, the marginal function has the same structure but the functions p, q are

$$p_\alpha = \alpha\, a \cosh\frac{a}{2} + \sinh\frac{a}{2}$$

$$\begin{aligned} q_{\alpha,\lambda} = \sinh\frac{c}{2}\Big[&(3b\lambda + d\alpha)\cosh e_p - (3b\lambda - d\alpha)\cos e_m + \\ &+ (\sqrt{3}e_m + (2b\alpha\lambda + 1)e_p)\sinh e_p - (\sqrt{3}e_p - (2b\alpha\lambda + 1)e_m)\sin e_m\Big] + \\ &- c\cosh\frac{c}{2}\Big[(b\alpha\lambda - 1)\cos e_m - (b\alpha\lambda - 1)\cosh e_p + \\ &- \alpha(\sqrt{3}e_m - e_p)\sinh e_p + \alpha(\sqrt{3}e_p + e_m)\sin e_m\Big] \end{aligned} \tag{7.11}$$

[1] Wolfram Research Inc.: Mathematica Version 7.

Remark 7.6. *To provide a homogeneous treatment, we do not distinguish the cases in which c is real or purely imaginary. Given the fact that the problem is real-defined, this implies that the functions we obtained can be either real valued or purely imaginary valued. In fact, function* (7.11) *is purely imaginary valued in the region $b > a^2$.*

Cases in which $\alpha = \beta$

It is easy to observe that, whenever $\alpha = \beta$, the marginal function (7.10) assumes the form $(C^2 - R^2)p_\alpha(a)q_{\lambda,\alpha}(a, R^2 - C^2)$, and is hence a function of a and $R^2 - C^2$ only, see Joseph (1976). This is the direct consequence of the fact that equations (7.5) can be recast as a system of equations for the fields $W, \Phi = \Gamma - \Theta, \Psi = \Gamma + \Theta$

$$\begin{cases} (D^2 - a^2)^2 W - a^2\Phi = 0 \\ (D^2 - a^2)\,\Phi + (C^2 - R^2)W = 0, \\ (D^2 - a^2)\,\Psi + (C^2 + R^2)W = 0, \end{cases}$$

When $\alpha = \beta$, also the boundary conditions become functions of W, Φ, Ψ. Solutions of the system can be found solving its first two equations. Hence, existence conditions depend on $a, R^2 - C^2$ only, and are those of the simple Bénard problem. The extreme cases are listed below.

A. Rigid boundaries ($\lambda = 0$) and fixed temperature and concentration ($\alpha = \beta = 0$): the marginal equations for the even and odd case are respectively

$$c \tanh \frac{c}{2} = \frac{(\sqrt{3}e_p - e_m)\sin e_m + (\sqrt{3}e_m + e_p)\sinh e_p}{\cos e_m + \cosh e_p},$$

$$c \coth \frac{c}{2} = \frac{(\sqrt{3}e_p - e_m)\sin e_m - (\sqrt{3}e_m + e_p)\sinh e_p}{\cos e_m - \cosh e_p}.$$

B. Stress–free boundaries ($\lambda = \infty$) and fixed temperature and concentration flows ($\alpha = \beta = 0$): the marginal equations for the even and odd case are respectively $\cosh[c/2] = 0$ and $\sinh[c/2] = 0$.

C. Rigid boundaries ($\lambda = 0$) and fixed temperature and concentration flows ($\alpha = \beta = \infty$): the marginal equations for the even and odd case are respectively

$$\frac{d}{c}\coth\frac{c}{2} = \frac{(e_m + \sqrt{3}e_p)\sin e_m + (\sqrt{3}e_m - e_p)\sinh e_p}{\cos e_m - \cosh e_p},$$
$$\frac{d}{c}\tanh\frac{c}{2} = \frac{(e_m + \sqrt{3}e_p)\sin e_m + (\sqrt{3}e_m - e_p)\sinh e_p}{\cos e_m - \cosh e_p}.$$

D. Stress-free boundaries ($\lambda = \infty$) and fixed temperature and concentration flows ($\alpha = \beta = \infty$): the marginal equations for the even and odd case are respectively

$$\frac{c}{2}\tanh\frac{c}{2} = \frac{e_m\sin e_m - e_p\sinh e_p}{\cos e_m + \cosh e_p},$$
$$\frac{c}{2}\coth\frac{c}{2} = \frac{e_m\sin e_m + e_p\sinh e_p}{\cos e_m - \cosh e_p}.$$

Cases in which $\alpha \neq \beta$

The remaining relevant cases are those in which the boundary conditions on temperature and concentration are extremal but different. A quick overview of (7.10) shows that the exchange of α with β in the boundary conditions yields an exchange of R^2 with C^2 in the equation. We hence write only two cases, with the understanding that the other two can be obtained by substituting R^2/C^2 with C^2/R^2.

E. Rigid boundaries ($\lambda = 0$), fixed heat flow and fixed concentration ($\alpha = \infty$, $\beta = 0$)

$$\frac{R^2}{C^2}\left[c\tanh\frac{c}{2}[(e_m + \sqrt{3}e_p)\sin e_m + (\sqrt{3}e_m - e_p)\sinh e_p] - d(\cos e_m - \cosh e_p)\right] =$$
$$a\tanh\frac{a}{2}\left[(\sqrt{3}e_p - e_m)\sin e_m + (\sqrt{3}e_m + e_p)\sinh e_p - c\tanh\frac{c}{2}(\cos e_m + \cosh e_p)\right],$$

$$\frac{R^2}{C^2}\left[c\coth\frac{c}{2}[(\sqrt{3}e_m - e_p)\sinh e_p - (e_m + \sqrt{3}e_p)\sin e_m] + d(\cos e_m + \cosh e_p)\right] =$$
$$a\coth\frac{a}{2}[(e_m - \sqrt{3}e_p)\sin e_m + (\sqrt{3}e_m + e_p)\sinh e_p + c\coth\frac{c}{2}(\cos e_m - \cosh e_p)].$$

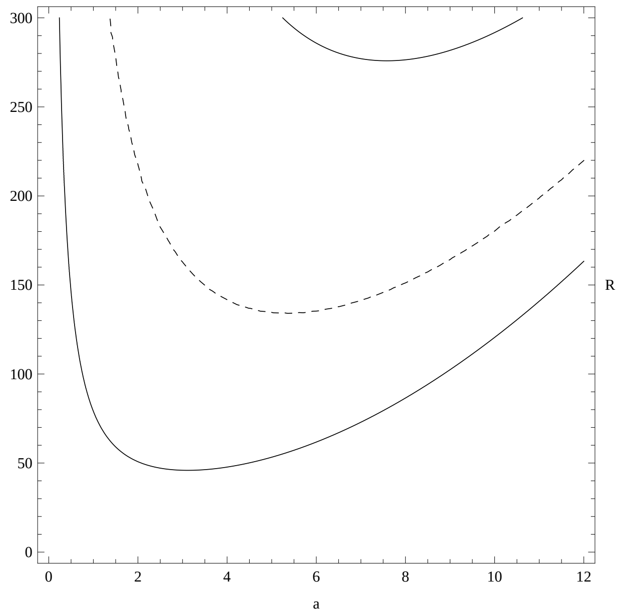


Figure 7.1: The marginal region associated to existence of even solutions (continuous lines) and odd solutions (dashed lines) with rigid, fixed temperature, and fixed solute boundary conditions, and with $C = 20$.

F. Stress–free boundaries ($\lambda = \infty$), fixed heat flow and fixed concentration ($\alpha = \infty$, $\beta = 0$)

$$3a\frac{R^2}{C^2}\tanh\frac{a}{2} = c\tanh\frac{c}{2} - 2\frac{e_m\sin e_m - e_p\sinh e_p}{\cos e_m + \cosh e_p},$$
$$3a\frac{R^2}{C^2}\coth\frac{a}{2} = c\coth\frac{c}{2} - 2\frac{e_m\sin e_m + e_p\sinh e_p}{\cos e_m - \cosh e_p}.$$

Plots of significant parity–preserving cases

The formulas above can be used to draw plots in a, R–space with fixed C. We use these plots to illustrate graphically some properties of the marginal regions. Figure 7.1 represents the typical structure of the marginal region when $\alpha \neq \infty$. The plot shows an alternation of non–intersecting curves associated to even and odd solutions.

Figure 7.2 shows that in the case $\alpha = \beta$ the marginal function depends on

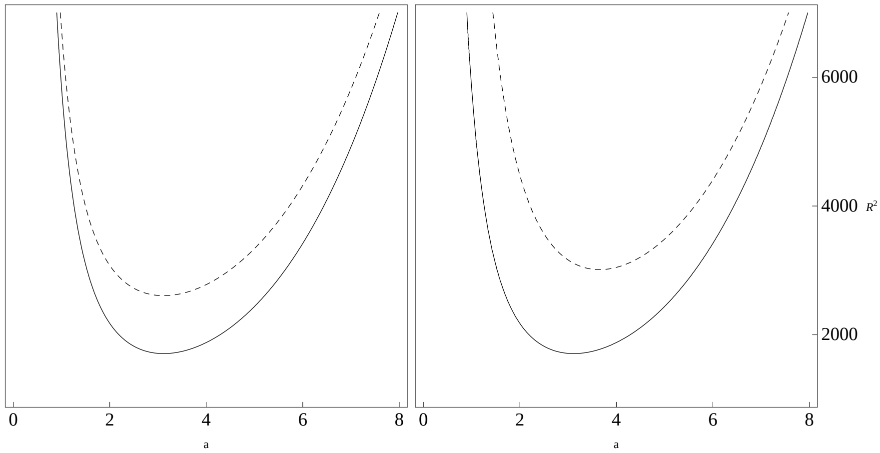


Figure 7.2: Lower branch of the marginal region in the space a, R^2 for $C = 0$ (continuous) and $C = 30$ (dashed). On the left, a case in which $\alpha = \beta$ shows the dependence on $R^2 - C^2$, on the right, a case in which $\alpha \neq \beta$.

a and $R^2 - C^2$ while, in the case $\alpha \neq \beta$, such property fails. In particular, only in the first case the marginal region in the a, R^2 space is vertically translated by changes of C and hence the a coordinate of the minimum is independent of C. The left frame corresponds to rigid boundaries, and fixed temperatures and solute concentrations, the right frame corresponds to rigid boundaries, fixed temperatures, and fixed solute fluxes. In both frames only the lowest branch, corresponding to even solutions, is shown for $C = 0$ and $C = 30$.

When the boundary condition is that of fixed heat flux, the marginal region significantly changes. In fact, the lowest branch of the marginal region approaches the R-axis at a finite value. Details of this case are given in Subsection 7.3.2.

For every boundary condition on velocity and solute except that of fixed solute flow, such limit turns out to be independent of C. We show this fact in Figure 7.3. We finally show, in Figure 7.4, that when the boundary conditions are of fixed heat flux and fixed solute flux, the asymptotic behavior is again dependent on C.

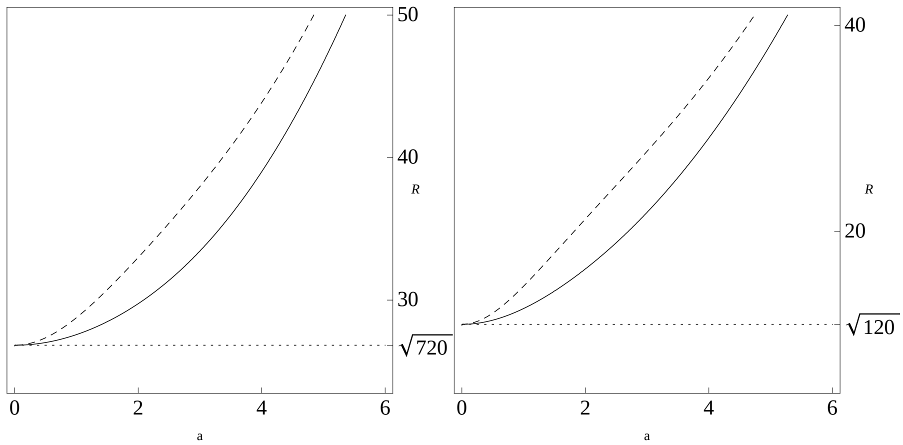


Figure 7.3: The asymptotic for $a \to 0$ of the marginal regions for fixed heat flow, fixed concentration, and $C = 10$ (continuous) or $C = 25$ (dashed). The left frame corresponds to rigid boundary conditions while the right frame corresponds to stress-free boundary conditions.

7.3.2 Application of the general formula: an investigation around $a = 0$

One of the striking features of marginal regions is that, under some boundary conditions (see Figures 7.3, 7.4), they have a finite asymptotic when a tends to zero. This phenomenon has already been described in Falsaperla and Lombardo (2009); Falsaperla and Mulone (2009, 2010); Falsaperla et al. (2010a). In this section we use the analytic expression obtained with the approach for non–symmetric boundaries to show that such asymptotic behavior takes place when and only when dealing with fixed heat flux at both boundaries.

The proof of this fact requires a Taylor expansion of the marginal function (7.8) around $a = 0$. A computation shows that the powers of the matrix A up to five have terms that are not divisible by a, while the power A^6 is divisible by a^2. It follows that a^{2i} divides A^k for $k \geq 6i$. Therefore, the coefficient of a^0 in a Taylor expansion of the marginal function can be computed using the

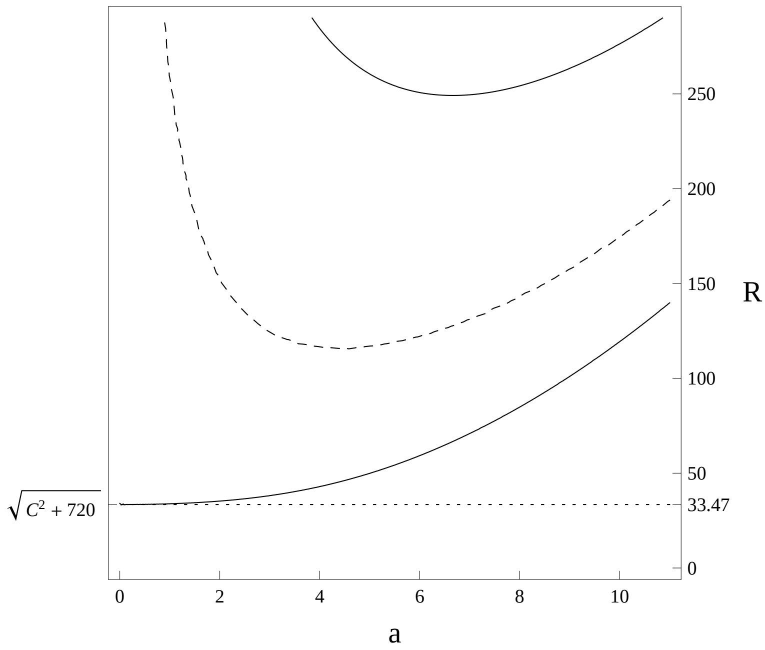


Figure 7.4: Asymptotic for $a \to 0$ in the rigid case, for fixed temperature flow and fixed concentration flow. In this picture, $C = 20$.

truncation of e^A to order five, which means that it suffice to compute

$$\det\left(P\left(I_8 + A + \frac{A^2}{2} + \cdots \frac{A^5}{5!}\right)B\right).$$

With a slight change in the convention on matrices P, B, that is writing boundary conditions (7.6) as convex combinations

$$(1-\lambda_\pm)DW \pm \lambda_\pm D^2W = (1-\alpha_\pm)\Theta \pm \alpha_\pm D\Theta = (1-\beta_\pm)\Gamma \pm \beta_\pm D\Gamma = 0,$$

the coefficient of a^0 turns out to be

$$(\alpha_-\alpha_+ - 1)(\beta_-\beta_+ - 1)(1 + 3\lambda_+ + 3\lambda_- 5\lambda_+\lambda_-)/12,$$

and it is obviously zero only when $\alpha_+ = \alpha_- = 1$ or $\beta_+ = \beta_- = 1$ (that correspond to fixed heat flux or fixed solute flow). Which means that in all other cases, the marginal region will not approach the R axis.

Assuming to be in the case of fixed heat flux, one has to investigate the coefficient of a^2 of the Taylor expansion. Such term can be computed using the truncation of the exponential e^A up to order eleven, and it turns out to be

$$\frac{\beta_-\beta_+ - 1}{8640}\left(R^2 - 720 + 8\left(R^2 - 270\right)(\lambda_- + \lambda_+) + 5\left(11R^2 - 720\right)\lambda_-\lambda_+\right).$$

The expression above can be zero if

$$R^2 = 720\frac{1 + 3(\lambda_+ + \lambda_-) + 5\lambda_+\lambda_-}{1 + 8(\lambda_+ + \lambda_-) + 55\lambda_+\lambda_-}, \tag{7.12}$$

which is, surprisingly, independent from C. So the solute does not have a stabilizing effect for any value of the gradient C^2. This independence was shown numerically in Falsaperla and Lombardo (2009); Falsaperla and Mulone (2009). The other possibility is that $\beta_+ = \beta_- = 1$. Assuming this, equating to zero the coefficient of a^4 one obtains an asymptotic limit which has the same expression of (7.12) with R^2 replaced by $R^2 - C^2$. The dependence of the limit on $R^2 - C^2$ is expected, as observed at the beginning of Section 7.3.1.

In the last case yet to analyze ($\beta_+ = \beta_- = 1$, but one of the $\alpha \neq 1$), the marginal region does not approach the a axis. In fact, the coefficient of a^2 is never zero. This fact would completely change if the fluid was salted form above.

Part II

Porous media

Chapter 8

Rotation

The problem of thermal convection in a rotating fluid layer is a well studied one with many applications, cf. Chandrasekhar (1961). The analogous problem in a rotating layer of a fluid saturated porous medium is one studied more recently, see Govender and Vadasz (2007); Malashetty and Heera (2008); Nield (1999); Sunil and Mahajan (2008b); Sunil et al. (2006); Vadasz (1997, 1998a,b), but nevertheless one with many mundane applications. When the boundary conditions on the temperature field are such that the temperature is fixed on the upper and lower horizontal planes then for zero inertia and Darcy's law, the linear instability and nonlinear stability problems are completely resolved and, in fact, coincide, cf. Straughan (2008); Vadasz (1998a).

The subject of this chapter is to analyze linear instability and global nonlinear stability for thermal convection in a rotating horizontal layer of saturated porous material but when the boundary conditions are not just these of Dirichlet type but Newton-Robin boundary conditions. This is not simply a mathematical exercise generalizing the work of Vadasz (1997, 1998a,b) or Straughan (2001, 2004, 2008). In applications, such as in geophysics, fixed temperature might be appropriate under cloudy conditions, but in strong sunlight prescription of heat flux is certainly necessary. Thus, Newton-Robin boundary conditions are likely to provide realistic forms for the temperature. We discover *a very interesting novel physical effect.* In particular, for Neumann boundary conditions on the temperature field, i.e. heat flux prescribed, we find that rotation has a pronounced effect on the critical wave number at which convection commences. For small rotation the critical wave number is zero. However, at a transition value of the rotation (Taylor number) instability switches and convection initiates with non-zero wave number. The exact values of the transition and the behavior of the critical Rayleigh number are calculated by a weakly nonlinear stability analysis. Equations for more general forms of convection in porous media are derived by Kannan and Rajagopal (2008); Rajagopal et al. (2009); Subramanian and Rajagopal (2007) and it will be interesting to study analogous effects in these theories.

We also include a global nonlinear stability analysis by means of an energy stability method, cf. Hill (2005); Sunil and Mahajan (2008a,b,c); Venkatasubramanian and Kaloni (2002). This is very important since we show the linear instability critical Rayleigh numbers and the global nonlinear stability Rayleigh numbers are the same. Thus, we can assert that a linearized instability analysis correctly predicts the physics of the onset of thermal convection.

8.1 Basic equations

Let us consider a layer of a porous medium bounded by two horizontal planes, $z = \pm d/2$, and rotating about a vertical axis z, heated from below. Let $d > 0$, $\Omega_d = \mathbb{R}^2 \times (-d/2, d/2)$ and $Oxyz$ be a cartesian frame of reference with unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, respectively. We assume that the Oberbeck-Boussinesq approximation is valid and the flow in the porous medium is governed by Darcy's law. The basic equations are:

$$\begin{cases} \nabla P = -\rho_f g \mathbf{k} - \dfrac{\mu_1}{K}\mathbf{v} - 2\dfrac{\rho_0}{\varepsilon}\boldsymbol{\omega} \times \mathbf{v} \\ \nabla \cdot \mathbf{v} = 0 \\ \dfrac{1}{M}\dfrac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = k\Delta T, \end{cases} \tag{8.1}$$

where $\rho_f = \rho_0[1 - \alpha(T - T_0)]$, $P = p_1 - \frac{1}{2}\rho_0[\boldsymbol{\omega} \times \mathbf{x}]^2$, p_1 is the pressure, $\boldsymbol{\omega} = \omega\mathbf{k}$ is the angular velocity field, and $\mathbf{x} = (x, y, z)$. The last term of $(8.1)_1$ is the *Coriolis acceleration*. We have denoted with $\varepsilon, \mathbf{v} = (U, V, W)$ and T the porosity, the seepage velocity and the temperature, respectively. The derivation of equations (8.1) may be found in Nield and Bejan (2006), pages 9, 24, 29. The quantities μ_1 and ρ_f are the viscosity and density of the fluid, and K denote the permeability of the medium. Further, g is the gravitational acceleration, k and M are respectively the effective thermal conductivity and the ratio of heat capacities, as defined in Nield and Bejan (2006).

Since the fluid is bounded by the two planes $z = \pm d/2$ we assume for the velocity field the following boundary condition

$$\mathbf{v} \cdot \mathbf{k} = 0 \qquad \text{on } z = \pm d/2$$

For the temperature, we use the following general boundary conditions

$$\begin{aligned} &\alpha_H(T_z + \beta)d + (1 - \alpha_H)(T_H - T) = 0, \qquad \text{on } z = -d/2 \\ &\alpha_L(T_z + \beta)d + (1 - \alpha_L)(T - T_L) = 0, \qquad \text{on } z = d/2, \end{aligned} \tag{8.2}$$

where $\alpha_H, \alpha_L \in [0, 1]$, $\beta > 0$, and $T_H = T_0 + \beta d/2$, $T_L = T_0 - \beta d/2$ are respectively an higher (T_H) and lower (T_L) temperature. Note that for $\alpha = 0$ or $\alpha = 1$, boundary conditions (8.2) imply respectively fixed temperature or fixed heat flux at a boundary. For $\alpha \in (0, 1)$ they are "finite conductivity" or Newton-Robin boundary conditions. Their form ensures that the basic solution for the thermal field is

$$T = -\beta z + T_0,$$

for any choice of α_H, α_L.

We introduce the non-dimensional quantities defined by

$$(\tilde{x}, \tilde{y}, \tilde{z}) \equiv \tilde{\mathbf{x}} = \frac{\mathbf{x}}{d}, \quad \tilde{t} = \frac{kM}{d^2}t, \quad \tilde{\mathbf{v}} = \frac{d}{k}\mathbf{v},$$

$$\tilde{T} = \frac{T - T_0}{T_H - T_L}, \quad \tilde{P} = \frac{K(P + \rho_0 g z)}{\mu_1 k}.$$

Omitting all tilde, the governing equations now take the following form

$$\begin{cases} \nabla P = \mathcal{R}^2 T\mathbf{k} - \mathbf{v} - \mathcal{T}\mathbf{k}\times\mathbf{v} \\ \nabla\cdot\mathbf{v} = 0 \\ \dfrac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T = \Delta T \end{cases} \tag{8.3}$$

where $\mathcal{T}^2$ and $\mathcal{R}^2$ are respectively the Taylor-Darcy number and the thermal Rayleigh number

$$\mathcal{T} = 2\frac{K\omega}{\varepsilon\nu}, \qquad \mathcal{R}^2 = \frac{\alpha g\beta dK}{\nu k},$$

and $\nu = \mu_1/\rho_0$ is the kinematic viscosity.

We shall study the stability of the motionless solution of (8.3)

$$m_0 = (\mathbf{v}, T, P)$$

given by

$$\mathbf{v} \equiv 0, \quad T = -z, \quad \nabla P = -\mathcal{R}^2 z\mathbf{k}.$$

Thus, the non-dimensional perturbation equations for a disturbance $(\mathbf{u}, \theta, p)$ to m_0 have the form:

$$\begin{cases} \nabla p = \mathcal{R}\theta\mathbf{k} - \mathbf{u} + \mathcal{T}\mathbf{u}\times\mathbf{k} \\ \nabla\cdot\mathbf{u} = 0 \\ \dfrac{\partial\theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = \mathcal{R}w + \Delta\theta, \end{cases} \tag{8.4}$$

(here we have made use of the transformation $\mathcal{R}\theta = \theta_1$ and we have omitted the subindex "$_1$").

The boundary conditions for the velocity field are

$$\mathbf{u}\cdot\mathbf{k} = 0, \quad \text{at} \quad z = -1/2, \quad z = 1/2.$$

For the temperature field we have

$$\begin{aligned} \alpha_H\theta_z - (1-\alpha_H)\theta = 0 \qquad & \text{at} \quad z = -1/2 \\ \alpha_L\theta_z + (1-\alpha_L)\theta = 0 \qquad & \text{at} \quad z = 1/2. \end{aligned} \tag{8.5}$$

The initial condition is given by

$$\theta(x, y, z, 0) = \theta_0(x, y, z).$$

We also assume that $\mathbf{u}, \theta, p$ satisfy plane tiling periodic boundary conditions in x and y and we denote by V the periodicity cell $V = [-1/2, 1/2] \times [0, 2\pi/a_1] \times [0, 2\pi/a_2]$, $a_1 > 0$, $a_2 > 0$ and $a = (a_1^2 + a_2^2)^{1/2}$.

To facilitate the analysis we take the third components of the *curl* and the *double curl* of $(8.4)_1$ to obtain:

$$\begin{cases} 0 = -\zeta + \mathcal{T} w_z \\ 0 = -\mathcal{R}\Delta^\star \theta + \Delta w + \mathcal{T}\zeta_z \\ \dfrac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla\theta = \mathcal{R} w + \Delta\theta, \end{cases} \tag{8.6}$$

where $\Delta^\star = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ and $\zeta = \mathbf{k} \cdot \nabla \times \mathbf{u}$ is the third component of the vorticity. An important aspect of this work is to establish global stability bounds for the nonlinear stability problem. In this regard it is advantageous to first consider the question of the principle of exchange of stabilities.

8.2 Exchange of stabilities

The linear equations associated to (8.6) are

$$\begin{cases} 0 = -\zeta + \mathcal{T} w_z \\ 0 = -\mathcal{R}\Delta^\star \theta + \Delta w + \mathcal{T}\zeta_z \\ \dfrac{\partial \theta}{\partial t} = \mathcal{R} w + \Delta\theta. \end{cases} \tag{8.7}$$

For a linearized analysis we consider solutions to (8.7) of the form

$$w(\mathbf{x}, t) = w(\mathbf{x})e^{\sigma t}, \quad \theta(\mathbf{x}, t) = \theta(\mathbf{x})e^{\sigma t}, \quad \zeta(\mathbf{x}, t) = \zeta(\mathbf{x})e^{\sigma t},$$

where the eigenvalue σ is a priori a complex number. Thus, the linearized perturbations satisfy the equations

$$\begin{cases} 0 = -\zeta + \mathcal{T} w_z \\ 0 = -\mathcal{R}\Delta^\star \theta + \Delta w + \mathcal{T}\zeta_z \\ \sigma\theta = \mathcal{R} w + \Delta\theta. \end{cases} \tag{8.8}$$

Now we prove that all the eigenvalues of (8.8) are real numbers, i.e. the *strong principle of exchange of stabilities* holds. We follow the lines of Straughan (2001), assuming that the quantities σ, w and θ are complex. We take the $\Delta^\star$ of eq. $(8.8)_3$, multiply it by the complex conjugate $\overline{\theta}$ of θ, and integrate over V

$$\sigma\|\nabla^\star\theta\|^2 = \mathcal{R}(\nabla^\star w, \nabla^\star\theta) - (\Delta^\star\Delta\theta, \theta),$$

here $(\cdot,\cdot)$ and $\|\cdot\|$ denote the scalar product and norm in the complex Hilbert space $L^2(V)$ and $\nabla^\star \equiv (\partial/\partial x, \partial/\partial y, 0)$. By taking into account boundary conditions (8.5), the last term in the previous equation can be rewritten as

$$(\Delta^\star\Delta\theta, \theta) = \|\nabla\theta_x\|^2 + \|\nabla\theta_y\|^2 + \tfrac{1-\alpha_L}{\alpha_L}\|\nabla^\star\theta\|^2_{\Sigma_+} + \tfrac{1-\alpha_H}{\alpha_H}\|\nabla^\star\theta\|^2_{\Sigma_-}$$

where Σ_+ and Σ_- are respectively $V \cap \{z = 1/2\}$ and $V \cap \{z = -1/2\}$, and $\|A\|_{\Sigma_\pm}$ denote the complex L^2-norm of a field A over the surfaces $\Sigma_\pm$, i.e. $\|A\|^2_{\Sigma_\pm} = \int_{\Sigma_\pm} |A|^2 \, d\Sigma$. We denote the general expression of the sum of the two surface integrals (which is a real non-negative quantity) by

$$S(A, \alpha_H, \alpha_L) = \frac{1-\alpha_L}{\alpha_L}\|A\|^2_{\Sigma_+} + \frac{1-\alpha_H}{\alpha_H}\|A\|^2_{\Sigma_-},$$

obtaining finally

$$\sigma\|\nabla^\star\theta\|^2 = \mathcal{R}(\nabla^\star w, \nabla^\star\theta) - \|\nabla\theta_x\|^2 - \|\nabla\theta_y\|^2 - S(\nabla^\star\theta, \alpha_H, \alpha_L). \tag{8.9}$$

Further, we multiply $(8.8)_2$ by $\overline{w}$ and, using also $(8.8)_1$, we derive

$$0 = \mathcal{R}(\nabla^\star\theta, \nabla^\star w) - \|\nabla w\|^2 - \mathcal{T}^2\|w_z\|^2. \tag{8.10}$$

By summing (8.9) and (8.10) we obtain

$$\begin{aligned}\sigma\|\nabla^\star\theta\|^2 &= \mathcal{R}[(\nabla^\star\theta,\nabla^\star w)+(\nabla^\star w,\nabla^\star\theta)]-\|\nabla w\|^2-\mathcal{T}^2\|w_z\|^2+\\ &\quad -\|\nabla\theta_x\|^2-\|\nabla\theta_y\|^2-S(\nabla^\star\theta,\alpha_H,\alpha_L).\end{aligned} \tag{8.11}$$

If we take the imaginary part of (8.11), and write σ as $\sigma_r+i\sigma_i$, we obtain

$$\sigma_i\|\nabla^\star\theta\|^2=0.$$

This implies $\sigma_i=0$, and thus the strong form of exchange of stabilities holds, unless $\|\nabla^\star\theta\|=0$, i.e. θ independent of x,y. In Straughan (2001) it is stated that $\|\nabla^\star\theta\|^2$ is always a non-null quantity for critical states, as demonstrated in Vadasz (1997). This condition, however, depends on the choice of the thermal boundary conditions, and we will see in the following that it is not satisfied for fixed heat fluxes at the boundaries. To consider the case $\|\nabla^\star\theta\|=0$, we observe that, by means of eq. $(8.8)_3$, this implies also $\|\nabla^\star w\|=0$. We start again from system (8.8) which, after the elimination of ζ, becomes

$$\begin{cases}(1+\mathcal{T}^2)w_{zz}=0\\ \sigma\theta=\mathcal{R}w+\theta_{zz}.\end{cases}$$

The first equation, because of the boundary conditions on w, implies $w=0$ and then

$$\sigma\theta=\theta_{zz}.$$

We multiply by $\overline{\theta}$ and integrate over V

$$\sigma\|\theta\|^2=-S(\theta,\alpha_H,\alpha_L)-\|\theta_z\|^2$$

and so $\sigma\in\mathbb{R}$, $\sigma\leq 0$.

8.3 Linear instability

Because of the validity of the principle of exchange of stabilities, at criticality, we have $\sigma=0$. Thus, we must solve the system

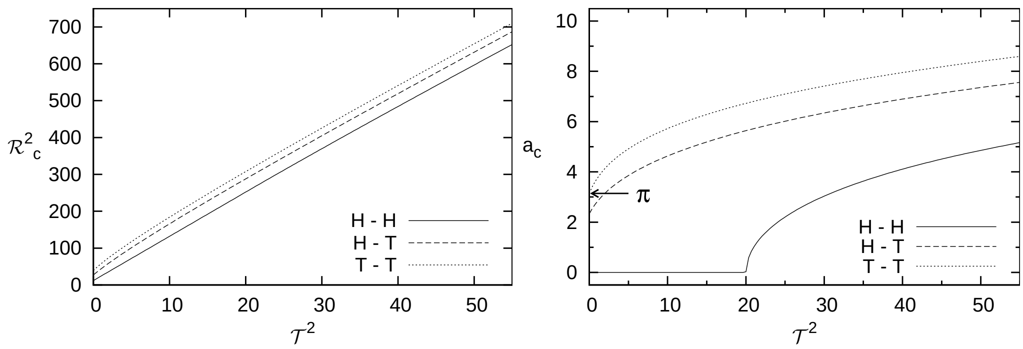


Figure 8.1: Critical Rayleigh number and critical wave number a_c as a function of $\mathcal{T}^2$ for different combination of thermal boundary conditions.

$$\begin{cases} -\zeta + \mathcal{T} w_z = 0 \\ -\mathcal{R}\Delta^\star\theta + \Delta w + \mathcal{T}\zeta_z = 0 \\ \mathcal{R}w + \Delta\theta = 0. \end{cases}$$

By eliminating ζ we obtain

$$\begin{cases} \Delta w + \mathcal{T}^2 w_{zz} - \mathcal{R}\Delta^\star\theta = 0 \\ \Delta\theta + \mathcal{R}w = 0. \end{cases}$$

We assume that the solutions are of the form

$$w = W(z)g(x,y), \quad \theta = \Theta(z)g(x,y), \qquad \text{where} \quad \Delta^* g + a^2 g = 0. \tag{8.12}$$

Substituting these periodic solutions into the system we then obtain the eigenvalue problem

$$\begin{cases} (1+\mathcal{T}^2)D^2 W - a^2 W + \mathcal{R}a^2\,\Theta = 0 \\ (D^2 - a^2)\Theta + \mathcal{R}\,W = 0 \end{cases} \tag{8.13}$$

with boundary conditions

$$\begin{aligned} &W = 0, && \text{at} \quad z = \pm 1/2, \\ &\alpha_H D\Theta - (1-\alpha_H)\Theta = 0, && \text{at} \quad z = -1/2, \\ &\alpha_L D\Theta + (1-\alpha_L)\Theta = 0, && \text{at} \quad z = 1/2, \end{aligned} \tag{8.14}$$

where "D" denotes the derivative along the z axis. For such boundary conditions it is not possible to obtain simple analytic solutions. We present in the following some numerical results obtained with the Chebyshev tau method described in Straughan (2001).

In Fig. 8.1 we show how the critical parameters depend on the rotation speed for different combinations of *fixed temperature* (T) and *fixed heat flux* (H) thermal boundary conditions. We observe that in all three cases rotation has a stabilizing effect on the system and that the maximum stability is obtained for thermostatic boundary conditions.

The critical wave number shows a qualitatively different behavior in the case of fixed heat fluxes, being equal to zero below a threshold of rotation speed.

Fig. 8.2 shows the dependency of a_c on $\mathcal{T}^2$ for Newton-Robin boundary conditions, i.e. for $\alpha_H, \alpha_L \in (0,1)$. We note that the same threshold effect observed for fixed heat fluxes is approximately present also for small values of α_H, α_L.

We examine then in detail the case of fixed heat fluxes at both boundaries, corresponding to the choice of $\alpha_H = \alpha_L = 1$ in the thermal boundary conditions. To better understand the behavior of a_c, in Fig. 8.3 we show the critical curves $\mathcal{R}^2(a)$ for fixed heat fluxes and different values of $\mathcal{T}^2$. We note that $\mathcal{R}^2$ is finite for $a = 0$, and the value $\mathcal{R}^2(0)$ increases regularly in all the range of values of $\mathcal{T}^2$. From Fig. 8.3 and Fig. 8.1b there is clearly a threshold in $\mathcal{T}^2$ such that instability changes from occurring at zero wave number to non-zero one. This is examined analytically by weakly nonlinear theory in the next section.

8.4 Nonlinear stability

The exact value of $\mathcal{R}^2(0)$ and the threshold $\mathcal{T}_a^2$ can be derived by an asymptotic expansion of the quantities appearing in system (8.13) in powers of a^2, i.e. via a weakly nonlinear analysis, cf. Roberts (1985).

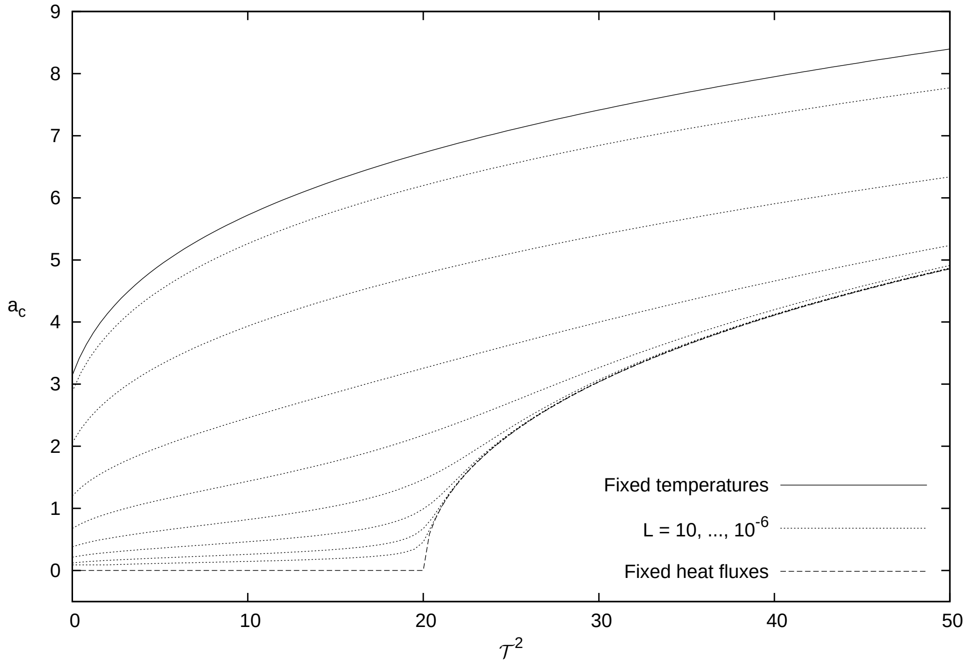


Figure 8.2: a_c as a function of $\mathcal{T}^2$ for thermal boundary conditions ranging from fixed heat fluxes ($\alpha = 1$, dashed line) to fixed temperatures ($\alpha = 0$, solid line). We consider here *symmetric* boundary conditions with $\alpha_H = \alpha_L \equiv \alpha$ and we have defined $L = (1 - \alpha)/\alpha$.

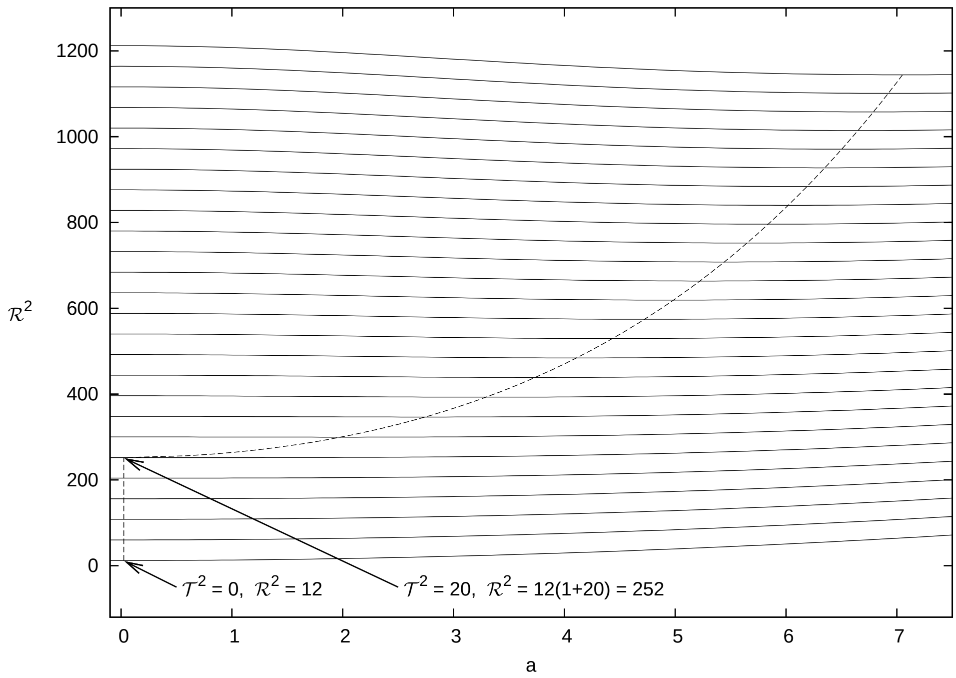


Figure 8.3: $\mathcal{R}^2$ as a function of a for fixed heat fluxes at both boundaries and selected values of $\mathcal{T}^2$. The continuous curves are computed for $\mathcal{T}^2 = 0, 4, 8, \ldots, 100$, from bottom to top. The dashed line is the locus of critical points $(a_c, \mathcal{R}_c^2)$, and so it intersects the continuous curves in their respective minima.

We assume for $W, \Theta, \mathcal{R}^2$ the following expansions in a^2

$$\begin{array}{lllllll} W(z) & = & W_0(z) & + & a^2 W_2(z) & + & a^4 W_4(z) & + \cdots , \\ \Theta(z) & = & \Theta_0(z) & + & a^2 \Theta_2(z) & + & a^4 \Theta_4(z) & + \cdots , \\ \mathcal{R}^2 & = & \mathcal{R}_0^2 & + & a^2 \mathcal{R}_2^2 & + & a^4 \mathcal{R}_4^2 & + \cdots . \end{array}$$

By standard analysis, cf. Roberts (1985), we find

$$\mathcal{R}_0^2 = 12\,(1 + \mathcal{T}^2), \qquad \mathcal{R}_2^2 = \frac{2}{35}(20 - \mathcal{T}^2).$$

Note that $\mathcal{R}_2^2$ is positive for $\mathcal{T}^2 < 20$, so $\mathcal{R}^2(a)$ has a (at least local) minimum for $a = 0$ in this range of values of $\mathcal{T}^2$. For $\mathcal{T}^2 > 20$, the point $a = 0$ becomes a local maximum, and so $\mathcal{R}^2(a)$ is minimum for some $a > 0$. This is in exact agreement with the numerical calculations.

We now construct a nonlinear energy stability analysis to verify how accurate the linear instability results are. Following the approach of Straughan (2001), we multiply eq. $(8.7)_3$ by θ, and integrate over the periodicity cell, obtaining

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|^2 = \mathcal{R}(w, \theta) + (\Delta\theta, \theta).$$

Again, boundary conditions (8.5) imply that the last term gives rise to a surface term,

$$(\Delta\theta, \theta) = -\|\nabla\theta\|^2 - S(\theta, \alpha_H, \alpha_L).$$

We take now eq. $(8.7)_2$, multiply it by w and integrate again over the cell, using also $(8.7)_1$, and obtain

$$0 = \mathcal{R}(\nabla^*\theta, \nabla^* w) - \|\nabla w\|^2 - \mathcal{T}^2\|w_z\|^2.$$

We can then form the identity

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|^2 = \mathcal{R}I - D, \tag{8.15}$$

where

$$I = (w, \theta) + (\nabla^*\theta, \nabla^* w)/\lambda,$$

$$D = \|\nabla\theta\|^2 + S(\theta, \alpha_H, \alpha_L) + (\|\nabla w\|^2 + \mathcal{T}^2\|w_z\|^2)/\lambda,$$

and $\lambda > 0$ is a parameter to be selected. We define $\mathcal{R}_E$ by

$$\frac{1}{\mathcal{R}_E} = \max_H \left(\frac{I}{D}\right), \tag{8.16}$$

where H is the space of admissible solutions. From (8.15) we find then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|^2 \leq -D\left(\frac{\mathcal{R}_E - \mathcal{R}}{\mathcal{R}_E}\right). \tag{8.17}$$

For $\alpha_H, \alpha_L \in (0,1]$, we can write the following chain of inequalities

$$D \geq \|\nabla\theta\|^2 + S(\theta, \alpha_H, \alpha_L) \geq c_\alpha c_0 \|\theta\|^2,$$

where

$$c_\alpha = \min\left\{1, \frac{1-\alpha_H}{\alpha_H}, \frac{1-\alpha_L}{\alpha_L}\right\}$$

and c_0 is the constant appearing in the Friedrichs' inequalities

$$\|\theta\|^2 \leq \|\theta\|^2_{W^{1,2}} \leq \frac{1}{c_0}\left(\|\nabla\theta\|^2 + \int_{\Sigma_\pm} |\theta|^2 \, d\Sigma\right).$$

So, for $\mathcal{R} < \mathcal{R}_E$, and α_H, α_L not both equal to 1, (8.17) implies $\|\theta(t)\| \to 0$ at least exponentially. From eq. $(8.4)_1$, multiplying by $\mathbf{u}$ and integrating over V, we have also

$$\|\mathbf{u}\|^2 = \mathcal{R}(\theta, w), \quad \text{which implies} \quad \|\mathbf{u}\| \leq \mathcal{R}\|\theta\|$$

so even $\|\mathbf{u}(t)\|$ is guaranteed to go to zero exponentially. To determine the stability threshold we need to solve the variational problem (8.16). It is important to note the the contribution of the surface term S appearing in D does cancel out in the evaluation of the Euler-Lagrange equations of problem (8.16), so we obtain the same E-L equations of Straughan (2001):

$$\begin{aligned} &\mathcal{R}_E(\lambda\theta - \Delta^*\theta) + 2(\Delta w + \mathcal{T}^2 w_{zz}) = 0 \\ &\mathcal{R}_E(\lambda w - \Delta^* w) + \ \lambda\Delta\theta = 0 \end{aligned}$$

Taking the double curl of the first equation, and assuming again solutions of the form (8.12), we obtain finally the system for W and Θ

$$\begin{cases} 2[(1+\mathcal{T}^2)D^2 - a^2]W + \mathcal{R}_E(\lambda + a^2)\,\Theta = 0 \\ 2\lambda(D^2 - a^2)\Theta + \mathcal{R}_E(\lambda + a^2)\,W = 0, \end{cases}$$

with boundary conditions (8.14). We note that the previous system reduces exactly to the system of linear instability (8.13) for $\lambda = a^2$ and then $\mathcal{R} = \mathcal{R}_E$ for this choice of λ (this is obviously confirmed by the numerical optimization of $\mathcal{R}_E$ w.r.t. λ). We can then affirm that the critical Rayleigh number obtained by a linear analysis is the real threshold of stability.

The last point is very important. It shows that the linear instability and the global nonlinear stability boundaries are the same. Thus, sub-critical instabilities will not arise and a linear instability analysis correctly captures the physics of the onset of convection.

Chapter 9

Binary fluid

Double diffusive convection in porous layers has many applications (Mulone and Straughan, 2006; Nield and Bejan, 2006; Straughan, 2008).

Here we consider the problem of a porous layer heated and salted from below. The motionless basic state has then a positive concentration gradient, having a stabilizing effect, and instability thresholds for this system have been studied for fixed temperatures and concentrations of mass. Here we consider more general boundary conditions on temperature and solute, in the form of Robin boundary conditions. As seen in Chapter 8, in the limit case of fixed heat fluxes the wavelength of the critical periodicity cell tends to infinity. In this work we investigate the influence of the solute field on this long-wavelength phenomenon, with the striking result that, as was the case for the Bénard system (see Chapter 4), the critical parameters become *totally independent* from the solute field.

9.1 Equations for a binary mixture

We assume that the layer has the same geometry used in the previous chapter and that the flow in the porous medium is governed by Darcy's law. Moreover we assume that the Oberbeck-Boussinesq approximation is valid with the same formal dependency of density on the temperature and the concentration of solute.

Under these assumptions, we follow the derivation of Mulone and Straughan (2006) (see also Nield and Bejan (2006); Straughan (2008)). Since the fluid is bounded by the two planes $z = \pm d/2$ we assume for the velocity field the boundary condition $\mathbf{k} \cdot \mathbf{v} = 0$ on $z = \pm d/2$. For the temperature and concentration fields, we use the same boundary conditions (4.1), (4.2) described previously.

We introduce the non-dimensional quantities defined in Straughan (2008),

$$(\tilde{x}, \tilde{y}, \tilde{z}) \equiv \tilde{\mathbf{x}} = \frac{\mathbf{x}}{d}, \quad \tilde{t} = \frac{kM}{d^2} t, \quad \tilde{\mathbf{v}} = \frac{d}{k} \mathbf{v},$$

$$\tilde{T} = \frac{T - T_0}{T_H - T_L}, \quad \tilde{C} = \frac{C - C_0}{C_H - C_L}, \quad \tilde{p} = \frac{K(p/\rho_0 + gz)}{\nu \kappa_T}.$$

Omitting all tilde, the governing equations now take the following form obtaining the following equations

$$\begin{cases} \nabla p = (\mathcal{R}^2 \, T - \mathcal{C}^2 \, C)\mathbf{k} - \mathbf{v}, \quad \nabla \cdot \mathbf{v} = 0, \\ T_t + \mathbf{v} \cdot \nabla T = \Delta T, \\ \epsilon \mathrm{Le} \, C_t + \mathrm{Le} \, \mathbf{v} \cdot \nabla C = \Delta C, \end{cases}$$

where

$$\epsilon = \varepsilon M, \quad \mathcal{R}^2 = \frac{\alpha_T \beta_T g d^2 K}{\nu \kappa_T}, \quad \mathcal{C}^2 = \frac{\alpha_C \beta_C g d^2 K}{\nu \kappa_T}$$

are the normalized porosity, and the thermal and solute Rayleigh numbers. Finally, $\mathrm{Le} = \kappa_T / \kappa_C$ is the Lewis number. We note that the two Rayleigh numbers definitions are slightly different from (4.5), but we retain the same symbols for ease of writing.

We shall study the stability of the motionless solution of (8.3) given by

$$\mathbf{v} \equiv 0, \quad T = -z, \quad C = -z, \quad \nabla p = (-\mathcal{R}^2 + \mathcal{C}^2) z \mathbf{k}. \tag{9.1}$$

In this way we obtain the following non-dimensional perturbation equations for a disturbance $(\mathbf{u}, \vartheta, \gamma, p_1)$ to (9.1)

$$\begin{cases} \nabla p_1 = (\mathcal{R}\,\vartheta - \mathrm{Le}\,\mathcal{C}\,\gamma)\mathbf{k} - \mathbf{u}, \\ \nabla \cdot \mathbf{u} = 0 \\ \vartheta_t + \mathbf{u} \cdot \nabla \vartheta = \mathcal{R} w + \Delta \vartheta, \\ \epsilon\,\mathrm{Le}\,\gamma_t + \mathrm{Le}\,\mathbf{u} \cdot \nabla \gamma = \mathcal{C} w + \Delta \gamma, \end{cases} \tag{9.2}$$

(quantities $\mathcal{R}$ and $\mathcal{C}$ appearing in Eq. 11 of Mulone and Straughan (2006) are respectively equal to quantities $\mathcal{R}^2$ and $\mathcal{C}^2$ used in the previous equation, and the changes of variables $\mathcal{R}\vartheta \to \vartheta, \mathcal{C}\gamma \to \gamma$ where performed). The quantities appearing in (9.2) have the same meaning of those used in the previous paragraph, $\mathrm{Le} = \kappa_T/\kappa_C$ is the Lewis number and ϵ is the normalized porosity (Mulone and Straughan, 2006).

9.2 Linear instability equations

We follow the standard analysis of Chandrasekhar (1961), applying twice the curl operator to the first equation. We then consider only the linear terms of the resulting systems and obtain

$$\begin{cases} 0 = \mathcal{R}\Delta^* \vartheta - \mathrm{Le}\mathcal{C}\Delta^* \gamma - \Delta w \\ \vartheta_t = \mathcal{R} w + \Delta \vartheta, \\ \epsilon \mathrm{Le} \gamma_t = \mathcal{C} w + \Delta \gamma. \end{cases} \tag{9.3}$$

where $\Delta^\star = \partial^2/\partial x^2 + \partial^2/\partial y^2$. We assume, as usual, that the perturbation fields are sufficiently smooth, and that they are periodic in the x and y directions (this is not a restriction, see Straughan (2004)). We denote by $a = (a_x^2 + a_y^2)^{1/2}$ the wave number. We search then solutions of the systems in the form

$$\begin{aligned} w &= W(z) \exp\{\, i\,(a_x x + a_y y) + p\,t\} \\ \vartheta &= \Theta(z) \exp\{\, i\,(a_x x + a_y y) + p\,t\} \\ \gamma &= \Gamma(z) \exp\{\, i\,(a_x x + a_y y) + p\,t\} \end{aligned} \tag{9.4}$$

for fields w, ϑ, γ, where $p = \sigma + i\tau$ is a complex constant. By substituting expressions (9.4) in (9.3) we obtain the following ODE system for the perturbation fields W, Θ, Γ

$$\begin{cases} 0 = (D^2 - a^2)W + \mathcal{R}a^2\,\Theta - \mathrm{Le}\mathcal{C}a^2\Gamma \\ p\,\Theta = (D^2 - a^2)\Theta + \mathcal{R}\,W, \\ p\,\epsilon\mathrm{Le}\,\Gamma = (D^2 - a^2)\Gamma + \mathcal{C}W. \end{cases}$$

where "D" represents the derivation along z. In this new variables, the hydrodynamic, thermal and solute boundary conditions become

$$\begin{array}{lll} \text{on } z = -1/2 & \alpha_H D\Theta - (1-\alpha_H)\Theta = 0, & \gamma_H D\Gamma - (1-\gamma_H)\Gamma = 0, \\ \text{on } z = 1/2 & \alpha_L D\Theta + (1-\alpha_L)\Theta = 0, & \gamma_L D\Gamma + (1-\gamma_L)\Gamma = 0. \end{array}$$

When the principle of exchange of stabilities (PES) holds, a simplified form of the system is obtained (Chandrasekhar, 1961).

9.3 Results

Results for porous media are qualitatively very similar, with respect to the dependence on thermal BCs, and also in the limit case of fixed heat fluxes. We present only a graphic for the critical Rayleigh number, for a choice of the Lewis number Le $= 1$ and of the normalized porosity $\epsilon = 2$ such that overstability is present for fixed temperatures. The same comments made on Fig. 4.1 apply to Fig. 9.1. The critical Rayleigh number becomes *independent of the concentration gradient* for fixed heat fluxes. An asymptotic analysis for $a \to 0$ confirms, even in this case, that $\mathcal{R}$ is constant, $\mathcal{R}_c^2 = 12$, independently of $\mathcal{C}$. Also in this case we obtain *global stability* (w.r.t. the classical energy norm) for fixed heat fluxes and any solute gradient.

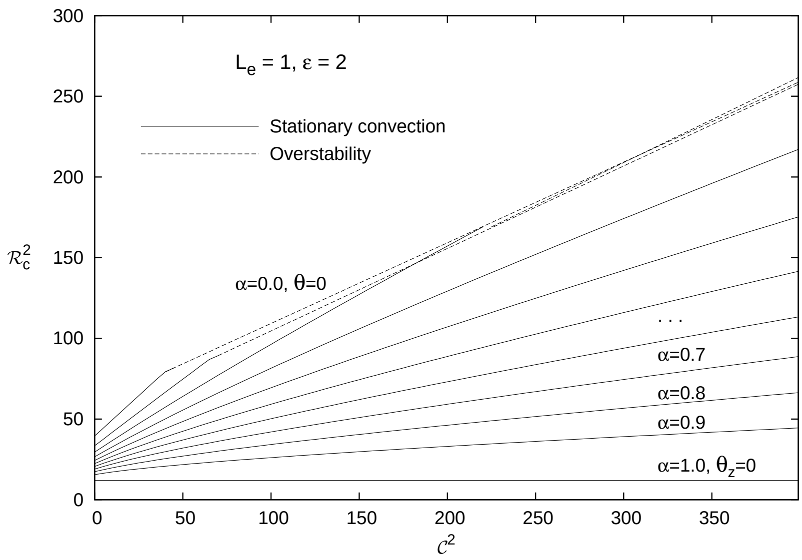


Figure 9.1: $\mathcal{R}_c^2$ as a function of $\mathcal{C}^2$ for thermal BCs going from fixed temperatures ($\alpha = 0$) to fixed heat fluxes ($\alpha = 1$).

Chapter 10

Rotation, including inertia effects

Thermal convection in a rotating fluid layer is a very well studied area of applied mathematics, cf. Chandrasekhar (1961); Mulone and Rionero (1994, 1997). The analogous problem in a rotating porous layer is one with many applications but is less well studied and analyses are more recent, see eg. Govender and Vadasz (2007); Malashetty and Heera (2008); Nield and Bejan (2006); Sunil and Mahajan (2008c); Sunil et al. (2006); Vadasz (1997). The prescribed boundary conditions on the temperature field, the instability and nonlinear stability problems are completely resolved when Darcy's law is adopted with zero inertia, cf. Straughan (2008); Vadasz (1998a).

Falsaperla and Mulone (2010) studied the problem of instability in a rotating layer of fluid when instead of simply prescribing the temperature field on the boundary, one employs Newton-Robin boundary conditions. For prescribed heat flux they found a striking result. For zero or low values of rotation they confirmed known results that instability occurs for zero wave number. However, they discovered that as the rotation rate is increased a critical value is reached and for a rotation rate beyond this convection commences for a non-zero wave number. Analyses of double-diffusion in a rotating layer employing similar boundary conditions are given by Falsaperla and Lombardo (2009) and by Falsaperla and Mulone (2009). Falsaperla et al. (2010a) analyzed the analogous problem in a Darcy porous medium with zero fluid inertia. They also discovered a critical rotation rate above which rotation commences with non-zero wave number. These writers explained this transition explicitly by means of a weakly nonlinear analysis.

In another development, Vadasz (1998a), discovered that the inclusion of fluid inertia in a rotating Darcy porous medium, with prescribed boundary temperatures, can lead to oscillatory instabilities (overstable convection) which are *not* present when inertia is zero. This is a striking result and because of this we now include inertia into the analysis of Falsaperla et al. (2010a). This is, however, a highly non-trivial extension. The mathematical problem is much more complicated and the physical picture is considerably richer. Given that we find (Falsaperla et al., 2010b) the inclusion of inertia can lead to much lower convection thresholds we believe this is a worthwhile area of convection to analyze. Experimental results on rotating porous convection should account for the possibility of inertia effects when being interpreted.

Even though the fluid inertia is often neglected in porous convection problems, continuum mechanical derivations of Darcy's law do show that, *a priori*, the fluid inertia should accounted for. Rajagopal (2009) derives general equations for non-isothermal motion in a porous medium by starting with the general equations of a mixture of a viscous fluid and an elastic material. He shows how models such as those of Darcy and Brinkman may be derived from this general theory. Derivation of Darcy equations with the fluid inertia term present are also given in Straughan (2008).

10.1 Basic equations

We consider a layer of porous medium bounded by two horizontal planes, $z = \pm d/2$. This layer rotates about a vertical axis z. The equations governing the velocity field and temperature in the layer $\{(x, y) \in \mathbb{R}^2\} \times \{z \in (-d/2, d/2)\}$, for the case where we are not far from the axis of rotation, are given by Vadasz (1998a), see also Nield and Bejan (2006).

The relevant equations incorporating fluid inertia, in the Oberbeck-Boussinesq approximation, are

$$\begin{cases} \frac{\rho_0}{\varepsilon}\frac{\partial v}{\partial t} = -\nabla P - \rho_f g\mathbf{k} - \frac{\mu_1}{K}\mathbf{v} - 2\frac{\rho_0}{\varepsilon}\boldsymbol{\omega} \times \mathbf{v} \\ \nabla \cdot \mathbf{v} = 0 \\ \frac{1}{M}\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = k\Delta T, \end{cases} \tag{10.1}$$

where $\boldsymbol{\omega} = \omega\mathbf{k}$ is the angular velocity of the layer, $\rho_f = \rho_0[1 - \alpha(T - T_0)]$ is the fluid density, linear in temperature. ρ_0 is the fluid density at a reference temperature T_0, and α is the thermal expansion coefficient of the fluid. The term ε is porosity, $\mathbf{v} = (U, V, W)$, g is gravity, $\mathbf{k} = (0, 0, 1)$, $\mathbf{v}$ being the seepage velocity. Also, k, M are the effective thermal conductivity and the ratio of fluid to solid heat capacities. On the boundaries $z = \pm d/2$ we suppose no throughflow

$$\mathbf{v} \cdot \mathbf{k} = 0 \quad \text{on } z = \pm d/2,$$

and for the temperature

$$\begin{aligned} &\alpha_H(T_z + \beta)d + (1 - \alpha_H)(T_H - T) = 0, \qquad && z = -d/2 \\ &\alpha_L(T_z + \beta)d + (1 - \alpha_L)(T - T_L) = 0, \qquad && z = d/2, \end{aligned}$$

both α_H, α_L being constants between 0 and 1, and where $T_z = \partial T/\partial z$. Additionally β is a positive constant with

$$T_H = T_0 + \frac{\beta d}{2}, \quad T_L = T_0 - \frac{\beta d}{2},$$

where T_H, T_L are constants with $T_H > T_L$. The limiting cases $\alpha = 0$ or $\alpha = 1$, yield prescribed temperature and prescribed heat flux, respectively. The basic motionless solution whose instability is to be investigated is given by

$$T = -\beta z + T_0.$$

Equation (10.1) may be non-dimensionalized with the scale

$$
(\tilde{x}, \tilde{y}, \tilde{z}) \equiv \tilde{\mathbf{x}} = \frac{\mathbf{x}}{d}, \quad \tilde{t} = \frac{kM}{d^2} t, \quad \tilde{\mathbf{v}} = \frac{d}{k} \mathbf{v},
$$

$$
\tilde{T} = \frac{T - T_0}{T_H - T_L}, \quad \tilde{P} = \frac{K(P + \rho_0 g z)}{\mu_1 k},
$$

$$
\mathrm{Va} = \frac{d^2 \mu_1 \varepsilon}{\rho_0 k M K}, \quad \mathcal{T} = 2\frac{K\omega\rho_0}{\varepsilon\mu_1}, \quad \mathcal{R}^2 = \frac{\alpha g \beta d K \rho_0}{\mu_1 k}.
$$

The tilde variables are non-dimensional, Va is the Vadasz number, $\mathcal{T}^2 = \mathrm{Ta}$ is the Taylor number, and $\mathcal{R}^2 = \mathrm{Ra}$ is the Rayleigh number.

Dropping the tilde, the non-dimensional version of equations (10.1) become

$$
\begin{cases}
\dfrac{1}{\mathrm{Va}} \dfrac{\partial \mathbf{v}}{\partial t} + \mathcal{T} \mathbf{k} \times \mathbf{v} + \mathbf{v} = -\nabla P + \mathcal{R}^2 T \mathbf{k} \\
\nabla \cdot \mathbf{v} = 0, \\
\dfrac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \Delta T.
\end{cases} \tag{10.2}
$$

Henceforth, our goal is to study the onset of convection and to this end we investigate the instability of the motionless solution to (10.2),

$$
\mathbf{v} \equiv 0, \quad T = -z, \quad \nabla P = -\mathcal{R}^2 z \, \mathbf{k}.
$$

The equations for a perturbation $(\mathbf{u}, \theta, p)$ to this basic solution which arise from (10.2) are then

$$
\begin{cases}
\dfrac{1}{\mathrm{Va}} \dfrac{\partial \mathbf{u}}{\partial t} = -\nabla p - \mathbf{u} - \mathcal{T} \mathbf{k} \times \mathbf{u} + \mathcal{R}^2 \theta \mathbf{k}, \\
\nabla \cdot \mathbf{u} = 0, \\
\dfrac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta = w + \Delta \theta.
\end{cases} \tag{10.3}
$$

The appropriate boundary conditions become

$$
\begin{aligned}
&\mathbf{u} \cdot \mathbf{k} = 0, && z = \pm 1/2 \\
&\alpha_H \theta_z - (1 - \alpha_H)\theta = 0, && z = -1/2 \\
&\alpha_L \theta_z + (1 - \alpha_L)\theta = 0, && z = 1/2
\end{aligned}
$$

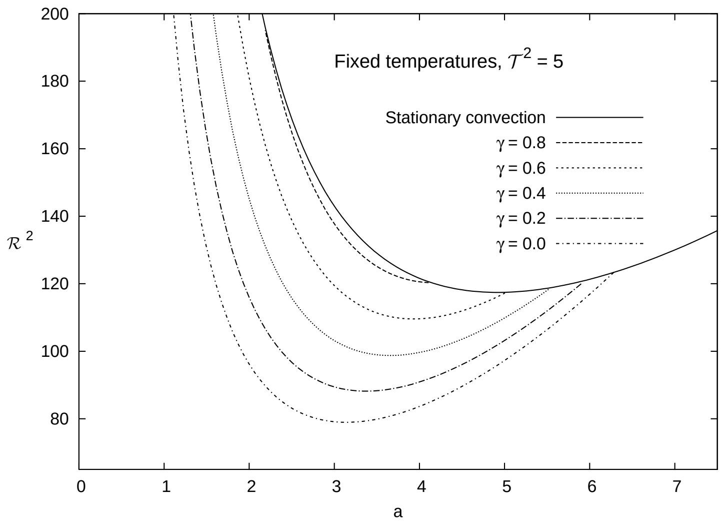


Figure 10.1: Critical Rayleigh number $\mathcal{R}^2$ as a function of the wave number a, at different values of the scaled Vadasz number $\gamma = \mathrm{Va}/\pi^2$. Temperature is fixed at the boundaries.

In addition, $\mathbf{u}, \theta, p$ are assumed periodic in the x, y directions.

To analyze linear instability we linearize (10.3) and take curl and curl curl of $(10.3)_1$. Retaining only the third components of the two equations so obtained, we are led to the system of equations

$$\begin{cases} \dfrac{1}{\mathrm{Va}}\dfrac{\partial\zeta}{\partial t} + \zeta = \mathcal{T} w_z, \\ \dfrac{1}{\mathrm{Va}}\dfrac{\partial}{\partial t}\Delta w + \Delta w + \mathcal{T}\zeta_z = -\mathcal{R}\Delta^\star\theta, \\ \dfrac{\partial\theta}{\partial t} = w + \Delta\theta, \end{cases} \tag{10.4}$$

where $\zeta = (\mathrm{curl}\,\mathbf{u})_3$, $w = \mathbf{u}_3$.

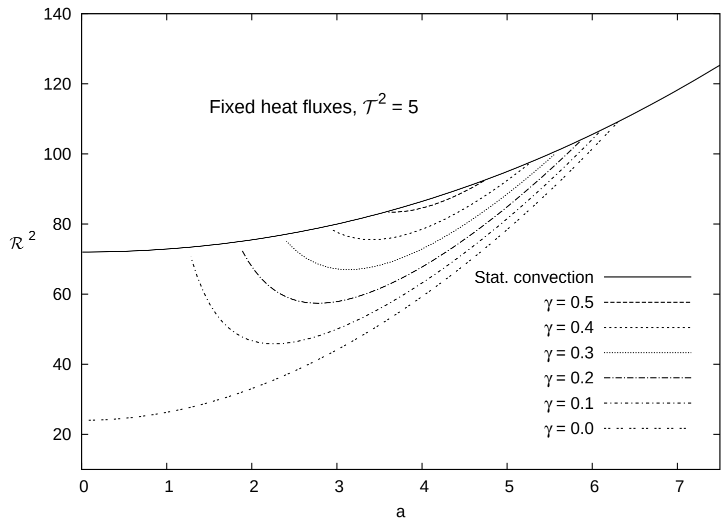


Figure 10.2: Same quantities of figure 10.1, for some values of the inertia coefficient γ. Heat fluxes are fixed at both boundaries.

10.2 Instability

To develop an instability analysis we convert equations (10.4) to an eigenvalue problem. Thus, we suppose w, ζ, θ can be written in the form

$$w = e^{\sigma t} W(z) f(x, y), \quad \zeta = e^{\sigma t} Z(z) f(x, y), \quad \theta = e^{\sigma t} \Theta(z) f(x, y),$$

where we mean a real Fourier series of such terms but because we only require instability one Fourier mode will suffice. The function $f(x, y)$ is a planform which tiles the plane, cf. Chandrasekhar (1961), §16.

If we denote by $D = d/dz$ and a is a wave number such that $\Delta^\star f = -a^2 f$, $\Delta^\star$ being $\Delta^\star = \partial^2/\partial x^2 + \partial^2/\partial y^2$, then equations (10.4) become

$$\begin{cases} \dfrac{\sigma}{\mathrm{Va}} Z + Z = \mathcal{T} DW \\ \dfrac{\sigma}{\mathrm{Va}} (D^2 - a^2) W + (D^2 - a^2) W = -\mathcal{T} DZ - \mathcal{R}^2 a^2 \Theta \\ \sigma \Theta = W + (D^2 - a^2) \Theta. \end{cases} \tag{10.5}$$

It is worth noting that we do not need boundary conditions on Z since we may eliminate Z from $(10.5)_1$ and $(10.5)_2$ to leave an eigenvalue problem which is nonlinear in σ, but the boundary conditions are sufficient to determine the solution.

To solve (10.5) in a practical manner we introduce the functions χ and Φ in such a way that $\chi = (D^2 - a^2)W$ and $\Phi = DZ$. Then, we may rewrite equations (10.5) as the system

$$\begin{cases} (D^2 - a^2)W - \chi = 0 \\ (D^2 - a^2)\Theta + W = \sigma\Theta \\ \chi + \mathcal{T}\Phi + \mathcal{R}a^2\Theta = -\dfrac{\sigma}{\mathrm{Va}}\chi \\ \mathcal{T}\chi + \mathcal{T}a^2 W - \Phi = \dfrac{\sigma}{\mathrm{Va}}\Phi. \end{cases} \tag{10.6}$$

System (10.6) is an eigenvalue problem for the eigenvalues $\{\sigma_n\}$ and is to be solved subject to the boundary conditions

$$\begin{array}{ll} W = 0, & z = \pm 1/2, \\ \alpha_H D\Theta - (1 - \alpha_H)\Theta = 0, & z = -1/2, \\ \alpha_L D\Theta + (1 - \alpha_L)\Theta = 0, & z = 1/2. \end{array} \tag{10.7}$$

The four boundary conditions (10.7) are sufficient to solve (10.6) since the last two equations of (10.6) can be regarded as identities. System (10.6,10.7) is solved numerically by means of the Chebyshev tau-QZ algorithm method, cf. Dongarra et al. (1996). Before presenting the numerical results we develop a weakly non linear analysis for (10.6,10.7) with $\alpha_H = \alpha_L = 1$, i.e. heat flux. The analysis works only for $Va = 0$, although the numerical results are also given for $Va \neq 0$. Thus, expand W, χ, Θ, Φ and $\mathcal{R}^2$ in a series in a^2, i.e.

$$\begin{array}{rcl} W &=& W_0 + a^2 W_2 + a^4 W_4 + \cdots \\ \chi &=& \chi_0 + a^2 \chi_2 + \cdots \\ \Theta &=& \Theta_0 + a^2 \Theta_2 + \cdots \\ \Phi &=& \Phi_0 + a^2 \Phi_2 + \cdots \\ \mathcal{R}^2 &=& \mathcal{R}_0^2 + a^2 \mathcal{R}_2^2 + \cdots . \end{array}$$

Next, solve the $O(1)$ and $O(a^2)$ problems recollecting the solution is generally complex, i.e. equating real and imaginary parts. Equations $(10.6)_3$ and

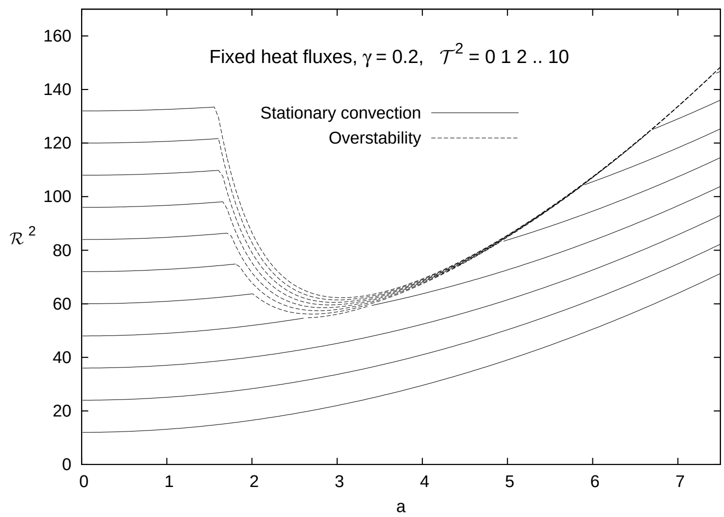


Figure 10.3: Critical curve $\mathcal{R}^2(a)$ for fixed heat fluxes and $\gamma = 0.2$. Taylor number varies from 0 to 10.

$(10.6)_4$ lead to the result

$$\mathcal{R}_0^2(1 + i\tau) = 12[(1 + i\tau)^2 + \mathcal{T}^2], \tag{10.8}$$

where $\sigma = r + i\tau$. Note also that $\mathcal{R}_0^2$ is just the value of $\mathcal{R}^2$ for $a = 0$. Then we obtain the two cases

$$\begin{aligned} \tau = 0 \quad &\rightarrow \qquad \mathcal{R}^2_{conv.} \equiv \mathcal{R}_0^2 = 12(1 + \mathcal{T}^2), \\ \tau \neq 0 \quad &\rightarrow \qquad \mathcal{R}^2_{over.} \equiv \mathcal{R}_0^2 = 24, \quad \tau^2 = \mathcal{T}^2 - 1. \end{aligned}$$

Since $\tau^2 > 0$, this implies overstability can only occur for $\mathcal{T}^2 > 1$. Moreover, for $\mathcal{T}^2 > 1$ we have $\mathcal{R}^2_{conv.} > \mathcal{R}^2_{over.}$, so overstability will be dominant.

10.3 Numerical results

In Fig. 10.1 we effectively repeat results of Vadasz (1998a). This shows that for fixed boundary temperatures and $\mathcal{T}^2 = 5$ then overstability does occur once $\gamma = \mathrm{Va}/\pi^2$ is less than a certain threshold (dependent on Va). This means that as Va decreases the system convects sooner, i.e. is less stable.

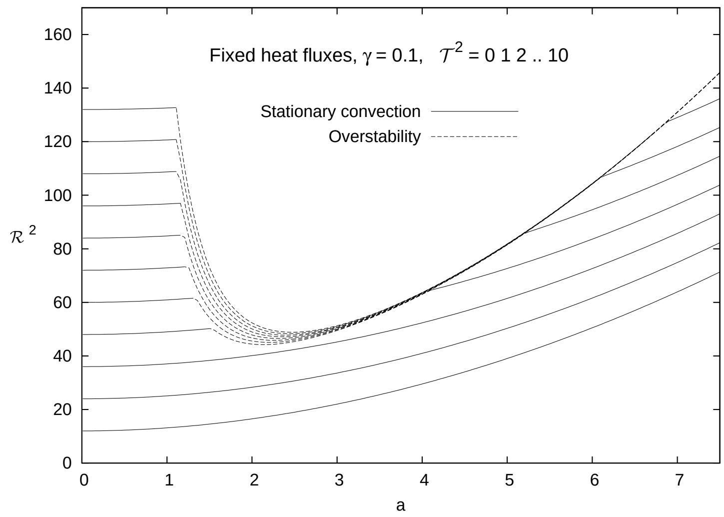


Figure 10.4: Critical curve $\mathcal{R}^2(a)$ for fixed heat fluxes and $\gamma = 0.1$. Taylor number varies from 0 to 10.

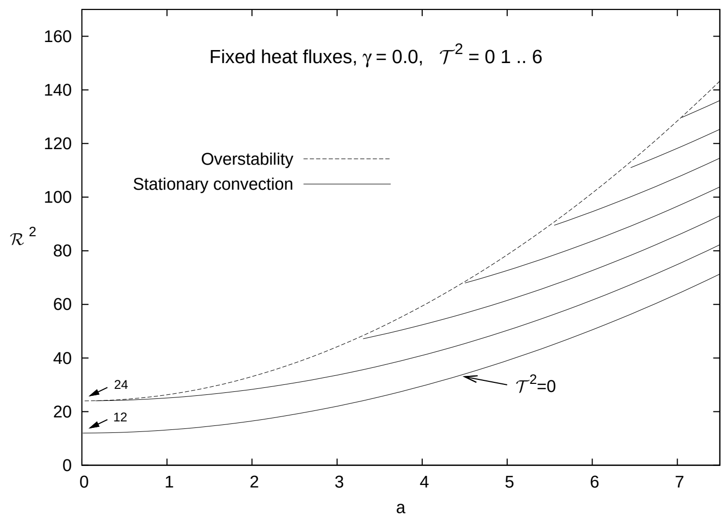


Figure 10.5: Critical curve $\mathcal{R}^2(a)$ for fixed heat fluxes and $\gamma = 0$. Taylor number varies from 0 to 6, from bottom to top.

Since the inertia coefficient is 1/Va this means increasing inertia leads to convection occurring more easily.

Fig. 10.2 presents the analogous situation when $\mathcal{T}^2 = 5$ but $\theta_z = 0$ on $z = \pm 1/2$. Again, we observe that increasing the inertia coefficient leads to oscillatory convection being the dominant mode of instability and a lowering of the convection threshold.

Figs. 10.3–10.5 show $\mathcal{R}^2$ against a for fixed heat flux boundary conditions and various values of $\mathcal{T}^2$. The quantity $\gamma = \mathrm{Va}/\pi^2$ decreases from 0.2 to 0.1 to 0 in figures 10.3–10.5.

Fig. 10.5 confirms the analytical result that overstability occurs only for $\mathcal{T}^2 > 1$ for fixed heat flux boundary conditions. The overstable part of the critical curve is the same for all the values of $\mathcal{T}^2$, for $\mathcal{T}^2 > 1$. Note that the theoretical values $\mathcal{R}^2 = 12$ for stationary convection at $\mathcal{T} = 0$, and $\mathcal{R}^2 = 24$ for overstability and $\mathcal{T}^2 \geq 1$, are confirmed numerically.

Figs. 10.6–10.8 show $\mathcal{R}^2$ against a with γ increasing, $\mathcal{T}^2 = 5$, but for various thermal boundary conditions, $\alpha = \alpha_H = \alpha_L$ increasing from 0 to 1. In particular, in figures 10.8 we see the $\mathcal{R}^2$ minimum is lower on some of the stationary convection curves. This corresponds to being in the region of stationary convection, $a = 0$, in figure 10.8b, with $\mathcal{T}^2 = 5$. Only for $\gamma = 0$ and fixed heat fluxes (Fig. 10.6) has the overstability curve a minimum for $a = 0$.

Finally, in figure 10.9 we show the dominant mode of instability in the $(\mathrm{Va}, \mathcal{T}^2)$ plane for $\theta_z = 0$ on the boundary. This curve is non-trivial to obtain and is obtained by computing $\mathcal{R}^2$ against a, as in figure 10.10, and varying $\mathcal{T}^2$ and Va. Figure 10.10 shows the situation where the stationary convection curve has the same minimum value as the oscillatory convection one for the sample values $\mathcal{T}^2 = 5, 10, 15, 20, 25$ and 30. For $\mathcal{T}^2 \leq 20$ the stationary convection minimum is obtained for $a = 0$. For $\mathcal{T}^2 > 20$ the minimum on the left is for a value of a with $a > 0$, as was shown in Falsaperla et al. (2010a) without inertia.

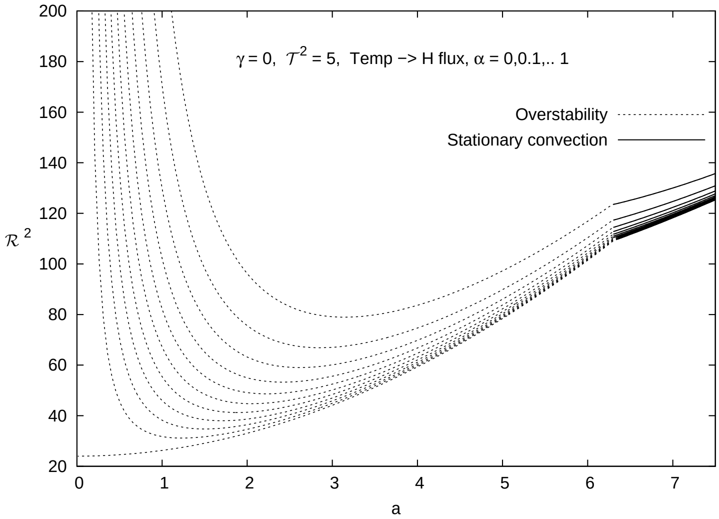


Figure 10.6: Critical curve $\mathcal{R}^2(a)$ for a sample value of the Taylor number $\mathcal{T}^2 = 5$, and boundary conditions going from fixed temperatures ($\alpha = 0$, top curve) to fixed heat fluxes ($\alpha = 1$, bottom curve). These curves are computed in the limit case $\gamma = 0$.

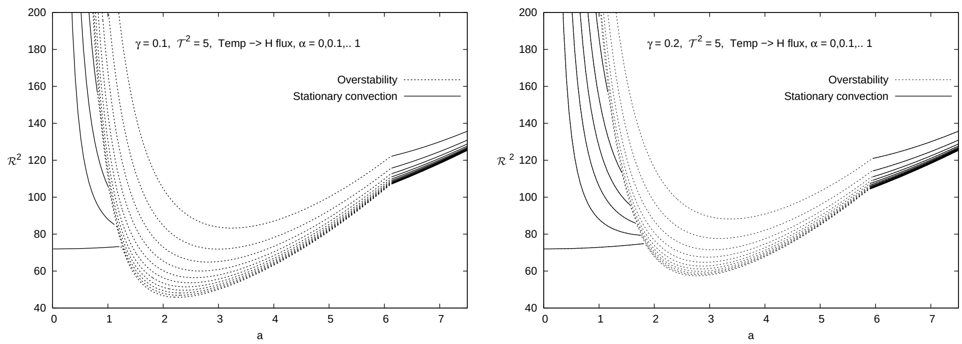


Figure 10.7: Same curves of Fig. 10.6, for $\gamma = 0.1$ and $\gamma = 0.2$.

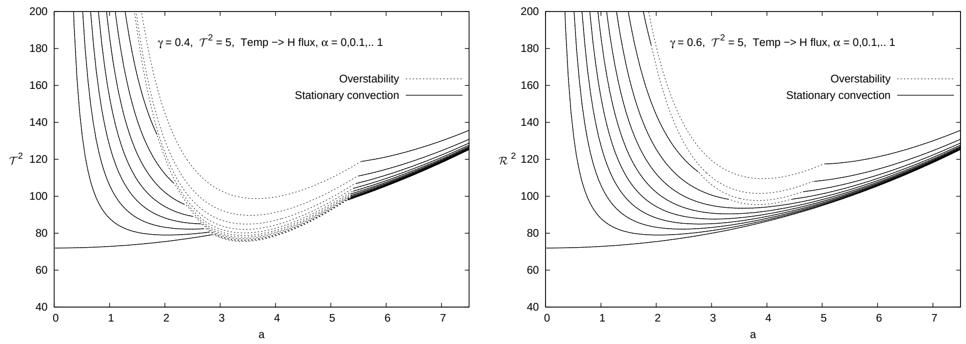


Figure 10.8: Same curves of Fig. 10.6, for $\gamma = 0.4$ and $\gamma = 0.6$.

10.4 Conclusions

We have analyzed a model for thermal convection in a layer of porous material which is rotating about an axis which is orthogonal to the layer. In particular, we have incorporated the fluid inertia coefficient into the analysis and we have analyzed the effect of this coefficient in the context of thermal boundary conditions of Newton-Robin type. In this way we extend the fundamental work of Vadasz (1998a) who used prescribed temperature boundary conditions, and the investigations of Falsaperla et al. (2010a) who neglected inertia.

The effect of inertia together with different thermal boundary conditions of Newton-Robin type leads to interesting physical behaviour. Indeed, the overall effect of including inertia is to lower the convection instability threshold. One also needs to have detailed knowledge of the thermal boundary conditions. However, if in an experiment lower instability threshold values are observed, lower than what might have been expected, then one ought to consider the inclusion of fluid inertia (often the inertial term is neglected in the engineering literature).

To obtain exact instability threshold one needs to consult the Rayleigh number curves presented here. However, figure 10.9 is of particular relevance. For fixed heat flux boundary conditions, this curve clearly delineates regions in $(\mathrm{Va}, \mathrm{Ta})$ space where overstability is the dominant instability mechanism, where stationary convection with zero wave number is observed, and where stationary convection with positive wave number will be found.

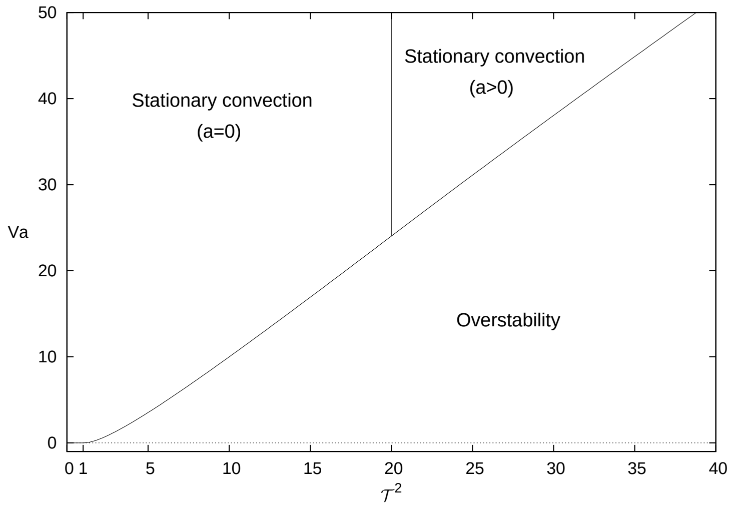


Figure 10.9: Mode of instability of the system for fixed heat fluxes, for different combinations of the Vadasz and Taylor numbers.

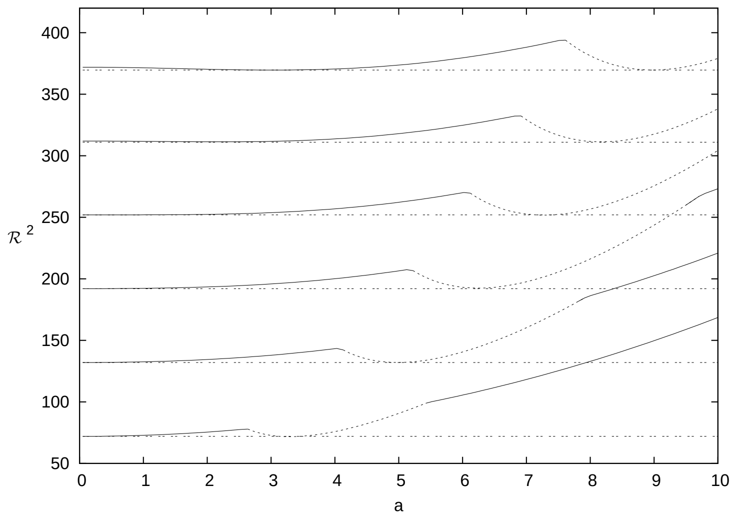


Figure 10.10: Critical curve $\mathcal{R}^2(a)$, for $\mathcal{T}^2 = 5, 10, 15, 20, 25, 30$ from bottom to top. The Vadasz number for each curve is such that the local minima in the stationary convection and overstable part of the curve have the same value of $\mathcal{R}^2$.

Chapter 11

Conclusions and open problems

In this thesis the effect of Newton-Robin (NR) boundary conditions on the temperature on layers of fluids and fluid saturating a porous medium is investigated. In all the considered cases, fixed temperatures have the most stabilizing effect, while for fixed heat fluxes we have the lowest values of the critical temperature gradient.

In Chapter 3 we investigate the rotating Bénard system. It is shown that for zero rotation, the principles of exchange of stabilities still holds for NR conditions. Thresholds of rotation for the appearance of overstability are derived, $\mathcal{T}^2 < \pi^4$ for RR and FF boundaries, and $\mathcal{T}^2 < \pi^4/4$ for RF boundaries ($\mathcal{T}^2$ is the non dimensional Taylor number, proportional to rotation). It is shown numerically that the result for FF boundaries is optimal, since in the limit of vanishing Prandtl number and for fixed heat fluxes, overstability appears for $\mathcal{T}^2 > \pi^4$. For fixed heat fluxes, the critical wave number, which is zero without rotation, is positive above some thresholds $\mathcal{T}_a^2$ of rotation, with $\mathcal{T}_a^2 \approx 77.32$, $\mathcal{T}_a^2 \approx 180.15$, $\mathcal{T}_a^2 \approx 1868.86$ for RF, FF, RR kinetic boundaries respectively. Rotation is confirmed to be stabilizing.

In Chapters 4 and 9 we study the stability of a layer of binary fluid, in the Bénard system and in porous media. In this case a striking result is obtained: in a transition from fixed temperatures to fixed heat fluxes the stabilizing effect of the solute is progressively reduced, and, for fixed heat fluxes, it is totally lost. This holds independently of the concentration gradient, and it is for this reason a very conterintuitive effect. An asymptotic analysis confirms nonetheless this result.

Chapters 5 and 6 study the joined effects of solute and rotation, and magnetic field and rotation, respectively. It is shown that, as is well known for fixed temperatures, solute and rotation continue to be independently stabilizing, while in some cases magnetic field and rotation are competitive, i.e. they are in same cases destabilizing. Same peculiar phenomena about the wave number appear also in these cases. In Chapter 5 is observed that the wave number is zero for low rotation speed, but in a region described very closely by the condition $\mathcal{T} > \mathcal{C}$ ($\mathcal{C}^2$ is the solute Rayleigh number, proportional to the solute gradient). Moreover, for any rotation number, the stabilizing effect of the solute is recovered. In Chapter 6 it is shown that vanishing wave numbers do not occur necessarily *below* a threshold rotation value, but also *above*. In the right panel of figure 6.1, for example, we note that the wave number is zero approximately for $\log \mathcal{Q}_1^2 \in (1.86, 2.64)$, and positive outside this interval. Here $\pi^2 \mathcal{Q}_1^2$ is the Chandrasekhar number, proportional to the external magnetic field.

An algebraic computation is developed in Chapter 7, showing that it is still possible to perform analytic computation even in the presence of different boundary conditions and stabilizing effects. The *analytical* results shown, contain as particular cases the classical results of Chandrasekhar (1961) for FF boundaries, but also those for the RR and RF cases. Moreover, all the intermediate cases, which correspond to finite slip effects, are included. Even if it is dubious that such computations can be further expanded, for example by including rotation effects, this technique is nonetheless useful, and will be probably applied again in future calculations.

In Chapter 3 convection in a rotating porous layer is investigated, and effects similar to those described in 3 are observed. In this case, for fixed heat fluxes, wave number is zero exactly for $\mathcal{T}^2 < 20$, and in this region of values of $\mathcal{T}$?, the critical Rayleigh number is given by $\mathcal{R}_c^2 = 12(1+\mathcal{T}^2)$. This latter results are derived by an asymptotic analysis of the system. This system is then studied in the case of non-negligible inertia effects in Chapter 10. Overstability, which was *never* present in the absence of inertia, now appears, and inertia is shown to be generally destabilizing. A detailed analysis of the region in the parameter space where stationary convection or overstability appear is performed in the case of fixed heat fluxes.

It will be interesting to extend all the above computations with a non linear stability analysis, which will permit to investigate the influence of boundary conditions on the basin of attraction of the basic motionless solution.

The stability of secondary convective flows will also be strongly influenced by thermal boundary conditions, and would be a phenomenon worthy of investigation.

The influence of boundary conditions on non stationary basic motions, could be also an interesting field of study, e.g. for a Couette or Poiseuille flow.

Finally, the study of convection in compressible fluids, modeled by the Navier-Stokes equations, in layers or more complicated geometries, could also lead to novel results for different thermal boundary conditions.

Appendix A

Asymptotic formulas

We report here asymptotic analyses of some of the eigenvalue problems described in the previous chapters, when the thermal boundary conditions are of prescribed heat flux.

A.1 Sample detailed computation

As an example of asymptotic analysis, we want to derive explicitly formula (10.8) of Chapter 10. We start from system (10.6), which we rewrite here

$$\begin{cases} (D^2 - a^2)W - \chi = 0 \\ (D^2 - a^2)\Theta + W = \sigma\Theta \\ \chi + \mathcal{T}\Phi + \mathcal{R}a^2\Theta = -\dfrac{\sigma}{\mathrm{Va}}\chi \\ \mathcal{T}\chi + \mathcal{T}a^2W - \Phi = \dfrac{\sigma}{\mathrm{Va}}\Phi, \end{cases} \tag{A.1}$$

subject to fixed heat flux boundary conditions

$$W = D\Theta = 0 \qquad \text{on } z = 0, 1.$$

Here, for ease of computation, we have translated the domain from $[-1/2, 1/2]$ to $[0, 1]$. We do not assume the validity of PES, so at criticality the eigenvalues σ are purely imaginary. We want also to study the limit case of large inertial effects, corresponding to $\mathrm{Va} \to 0$, so it is convenient to introduce the real quantity τ such that $\sigma/\mathrm{Va} = i\tau$. System (A.1) then becomes, for $\mathrm{Va} \to 0$,

$$\begin{cases} (D^2 - a^2)W - \chi = 0 \\ (D^2 - a^2)\Theta + W = 0 \\ \chi + \mathcal{T}\Phi + \mathcal{R}a^2\Theta = -i\tau\chi \\ \mathcal{T}\chi + \mathcal{T}a^2 W - \Phi = i\tau\Phi. \end{cases} \tag{A.2}$$

We want to study solutions of this system for $a \to 0$, and in particular we want some explicit expression for the critical curve $\mathcal{R}^2(a)$. So we assume that all fields and $\mathcal{R}^2$ can be expanded in powers of a^2 as follows

$$\begin{array}{rcccccl} W(z) &=& W_0(z) &+& a^2\, W_2(z) &+& a^4\, W_4(z) + \cdots \\ \chi(z) &=& \chi_0(z) &+& a^2\, \chi_2(z) &+& \cdots \\ \Theta(z) &=& \Theta_0(z) &+& a^2\, \Theta_2(z) &+& \cdots \\ \Phi(z) &=& \Phi_0(z) &+& a^2\, \Phi_2(z) &+& \cdots \\ \mathcal{R}^2 &=& \mathcal{R}_0^2 &+& a^2\, \mathcal{R}_2^2 &+& \cdots\ . \end{array}$$

If we substitute the above expressions into (A.2), and retain only the zeroth order terms, we obtain

$$\begin{cases} D^2 W_0 - \chi_0 = 0 \\ D^2\Theta_0 + W_0 = 0 \\ (1 + i\tau)\chi_0 + \mathcal{T}\Phi_0 = 0 \\ \mathcal{T}\chi_0 - (1 + i\tau)\Phi_0 = 0. \end{cases} \tag{A.3}$$

If we consider the last two equation, in the fields χ_0 and Φ_0, we see that the determinant of this sub-system is given by $\mathcal{T}^2 + (1+i\tau)^2$ or $\mathcal{T}^2 + 1 + 2i\tau - \tau^2$. This complex quantity can be equal to zero only for $\tau = 0$ and $\mathcal{T}^2 + 1 = 0$, but this is impossible since $\mathcal{T}$ is real. We have then $\Phi_0 = \chi_0 = 0$, and from the first (differential) equation of (A.3), and the boundary conditions $W_0(0) = W_0(1) = 0$, we get $W_0 = 0$. The remaining equation becomes $D^2\Theta_0 = 0$ which, because of the boundary conditions $D\Theta_0 = 0$, implies $\Theta = C$, where C is an arbitrary complex constant. We can assume simply $C = 1$, without loss of generality, and then our zeroth order solution is

$$W_0(z) = \chi_0(z) = \Phi_0(z) = 0, \quad \Theta_0(z) = 1.$$

We consider now the system obtained from (A.2), by retaining only the terms which are of second order in a,

$$\begin{cases} D^2W_2 - W_0 - \chi_2 = 0 \\ D^2\Theta_2 - \Theta_0 + \mathcal{R}^2 W_0 + \mathcal{R}_0^2 W_2 = 0 \\ (1+i\tau)\chi_2 + \Theta_0 + \mathcal{T}\Phi_2 = 0 \\ \mathcal{T}\chi_2 + \mathcal{T}W_0 - (1+i\tau)\Phi_2 = 0. \end{cases}$$

Note that in this system appear also some of the zeroth order functions, which we can then substitute, obtaining

$$\begin{cases} D^2W_2 - \chi_2 = 0 \\ D^2\Theta_2 - 1 + \mathcal{R}_0^2 W_2 = 0 \\ (1+i\tau)\chi_2 + \mathcal{T}\Phi_2 = -1 \\ \mathcal{T}\chi_2 - (1+i\tau)\Phi_2 = 0. \end{cases}$$

We consider again the last two equation, which are now a non-homogeneous system, and obtain

$$\chi_2 = -\frac{1+i\tau}{(1+i\tau)^2 + \mathcal{T}^2} \equiv A.$$

Then, by integrating the first equation, with $W_2(0) = W_2(1) = 0$, we find

$$W_2(z) = \frac{A}{2}(z^2 - z).$$

Finally, from the second equation

$$D^2\Theta_2(z) = 1 + \mathcal{R}_0^2 \frac{A}{2}(z^2 - z),$$

integrating once we obtain

$$D\Theta_2(z) = z + \mathcal{R}_0^2 \frac{A}{2}(\frac{z^3}{3} - \frac{z}{2}) + c,$$

where c is an other arbitrary quantity. Imposing then the boundary conditions on $D\Theta$ we get $c = 0$ and

$$\mathcal{R}_0^2 = -\frac{12}{A},$$

that is

$$(1 + i\tau)\mathcal{R}_0^2 = 12[(1 + i\tau)^2 + \mathcal{T}^2].$$

The real and imaginary part of this equation are

$$\begin{aligned}\mathcal{R}_0^2 &= 12(1 + \mathcal{T}^2 - \tau^2)\\ \tau\mathcal{R}_0^2 &= 24\tau,\end{aligned}$$

from which we can derive the final results of section 10.2.

A.2 Electrically conducting fluid

We consider system (6.1), describing the thermal instability of a rotating layer of electrically conducting fluid, subject to an external vertical magnetic field. We assume the validity of PES, and derive our formulas for the case of a non-rotating layer. Then, by setting $p = 0$ and $\mathcal{T} = 0$, we obtain

$$\begin{cases}(D^2 - a^2)^2W + D(D^2 - a^2)K - DZ - a^2\,\Theta = 0,\\ (D^2 - a^2)Z + DX = 0,\\ (D^2 - a^2)X + \mathcal{Q}^2\,DZ = 0,\\ (D^2 - a^2)K + \mathcal{Q}^2\,DW = 0,\\ (D^2 - a^2)\Theta + \mathcal{R}^2\,W = 0,\end{cases}$$

and we can then eliminate field K, obtaining

$$\begin{cases}(D^2 - a^2)^2W - \mathcal{Q}^2\,D^2W - DZ - a^2\,\Theta = 0,\\ (D^2 - a^2)Z + \mathcal{T}^2DW + DX = 0,\\ (D^2 - a^2)X + \mathcal{Q}^2\,DZ = 0,\\ (D^2 - a^2)\Theta + \mathcal{R}^2\,W = 0.\end{cases}$$

We consider the three combinations of rigid (R) and free (F) boundary conditions RR, RF and FF, non conducting boundaries, and fixed heat fluxes at both boundaries, so the system is subject to

$$\begin{aligned}&W = X = D\Theta = 0 && \text{on both boundaries,}\\ &D^2W = DZ = 0 && \text{on free boundaries,}\\ &DW = Z = 0 && \text{on rigid boundaries.}\end{aligned}$$

By expanding the fields and the Rayleigh number in powers of a^2, and after some lengthy calculations, we obtain the following expression of the critical Rayleigh number for $a \to 0$

$$\mathcal{R}_0^2 = \frac{12\mathcal{Q}^4(1-e^{\mathcal{Q}})}{12+6\mathcal{Q}+\mathcal{Q}^2-e^{\mathcal{Q}}(12-6\mathcal{Q}+\mathcal{Q}^2)}$$

for the RR case,

$$\mathcal{R}_0^2 = \frac{12\mathcal{Q}^5(1+e^{2\mathcal{Q}}(\mathcal{Q}-1)+\mathcal{Q})}{-24-24\mathcal{Q}+4\mathcal{Q}^3+\mathcal{Q}^4-24\,e^{\mathcal{Q}}(\mathcal{Q}^2-2)-e^{2\mathcal{Q}}(24-24\mathcal{Q}+4\mathcal{Q}^3-\mathcal{Q}^4)}$$

for the RF case, and

$$\mathcal{R}_0^2 = \frac{12\mathcal{Q}^5}{-12\mathcal{Q}+\mathcal{Q}^3+12\sinh(\mathcal{Q})-24\sinh(\mathcal{Q}/2)^2\tanh(\mathcal{Q}/2)}$$

for the FF case.

Expanding the above three results in a Taylor series in $\mathcal{Q}$ we get also the respective expressions

$$\mathcal{R}_0^2 = 720+\frac{120}{7}\mathcal{Q}^2-\frac{1}{49}\mathcal{Q}^4+O(\mathcal{Q}^6),$$
$$\mathcal{R}_0^2 = 320+\frac{320}{21}\mathcal{Q}^2-\frac{88}{3969}\mathcal{Q}^4+O(\mathcal{Q}^6),$$
$$\mathcal{R}_0^2 = 120+\frac{85}{7}\mathcal{Q}^2-\frac{5}{3528}\mathcal{Q}^4+O(\mathcal{Q}^6),$$

which confirm that these formulas have the correct limits for $\mathcal{Q} \to 0$, corresponding to the known values for the simple Bénard system of $720, 320, 120$. The magnetic field, as expected, has a stabilizing effect in all three cases, since the three values of $\mathcal{R}_0^2$ are increasing functions of $\mathcal{Q}$.

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