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VARIATIONAL INEQUALITIES AND NETWORKS FOR ORGAN TRANSPLANTS AND FOR HUMANITARIAN ORGANIZATIONS

PhD Thesis

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Introduction

The power of mathematics is exploited in almost all areas of research: physics, engineering, chemistry, economics, computing, sociology, medicine and so on. In particular, *applied mathematics* is certainly part of the discipline most widely used by everyone.

Among its various branches is the *Operations Research*, which deals with decision-making problems related to the operation of organized systems with the aim of determining decisions that optimize their performance. The Operations Research was born during the Second World War: some countries decided to apply mathematical models to problems related to military operations. The goal was to improve the performance of the war apparatus by not developing new weapons but improving the use of existing ones. For this purpose, the British Government set up a group of scientists from several disciplinary areas called “operational research section”. That term was replaced by the present one “operations research” when, in 1943, an organization which was similar to the British one was born in the United States, dealing with under water warfare, supply system and logistical support.

Over the years, numerous results have been achieved in the area of Operational Research thanks to the contribution of several scholars: J.L. von Neumann, who developed a series of models for studying economic grow thin conditions of competitive equilibrium, decision-making in a multidisciplinary environment, and how to run computing programs; P.M.S. Blackett and T.C. Koopmans studying models for logistic applications; G.B. Dantzig who contributed to the development of linear programming with the introduction of

a resolution operating method known as the "simplex method".
Around 1950 the first results in the area of network optimization algorithms began to appear and in the 1960s the researchers focused the problem of assessing the efficiency of solving decision-making algorithms. At the end of the twentieth century, the success of Operational Research was also due to the development of mathematical models, solving algorithms, and computer and network technologies.
The typical applications of the Operational Research are: *warehouse management*, in which, taking into account production costs, it is necessary to determine the optimal quantities to be retained for future use; *portfolio choices*, where you need to decide on the amount to be invested to maximize your earnings, knowing the initial financial resources and investment opportunities; *the plans for creating and managing activities for the design of a complex system*, in which you have to choose the starting times of the activities in accordance with their duration and the resources to be used in order to minimize the time and cost of realization; *the plan of production and industrial logistics*; and so on.
The decision making process which Operational Research deals with is to build a mathematical model of the system that represents it. This process begins with a system analysis, which involves the definition of its components, the parameters that characterize it, and the relationships that exist between them. Next, we identify the decision-making variables and the set of constraints of the problem. The mathematical model thus formed is solved by a method or algorithm. The obtained numerical results represent the optimal solution of the model, that is the optimal quantities of the decision to be taken. In fact, most decision problems are formulated as optimization problems whose purpose is to identify the best decisions to take to achieve the goal.
The choice of which mathematical tools to use is influenced by the type of model you want to study and the most appropriate approach. The most used tools are the dynamical systems, the variational inequalities, the game

theory, etc. In addition, in formulating the mathematical model, defining the network on which the entire model is based plays a central role. The network allows us to define the different levels of decision makers involved in the whole process and its flows. The network concept plays a central role in decision-making processes and its methods of analysis are applicable not only to physical networks such as transport networks, energy networks, production and logistics, but also to complex networks such as supply chains, financial networks, social networks, knowledge networks, and economic networks.

In this thesis we focus our attention on two mathematical models applied to two real situations, both studied with the theory of variational inequalities. The first network-based model describes the organ transplant system with the aim of minimizing the total costs associated with this process. We find the related optimality conditions and the variational inequality formulation. Some existence and uniqueness results as well as the Lagrange formulation are stated and some numerical examples are studied. The second mathematical model presented is a Generalized Nash Equilibrium model for post-disaster humanitarian relief, which extends the original model of Nagurney, Alvarez Flores, and Soylu (see [46]). We identify the network structure of the problem, with logistical and financial flows, and propose a variational equilibrium framework, which allows us to then formulate, analyze, and solve the model using the theory of variational inequalities. We then utilize Lagrange analysis and finally we illustrate the game theory model through a case study.

The thesis is structured as follows. In Chapter 1, we recall the concept of optimization problems and the two mathematical tools used to study the models presented above: the theory of variational inequalities and the Lagrange theory. Finally we present the main results on traffic network equilibrium. In Chapter 2, we present some mathematical models related to the transplant process existing in the literature. In Chapter 3 and 4 we analyze, respectively, the network model for minimizing the total organ transplant costs and the variational equilibrium network for humanitarian organizations in disaster relief. Finally, in Chapter 5 we present the conclusions.

Chapter 1

Theory and Fundamentals

In real life many problems, studied in different disciplines, consist to find the maximum value or the minimum value of an assigned amount. Specifically, you want to determine the maximum when that "amount" represents a benefit or a profit, the minimum when it represents a cost.
The discipline dealing with such problems takes the name of *Optimization Theory*. Among the various optimization problems are those where you need it determine the optimal values of a function, whose decision-making variables are subject to constraints expressed by equality and/or inequality. Such types of problems are the subject of so-called *mathematical programming*, term introduced by Robert Dorfman in 1949.

An optimization problem can be stated as follows: find $x = (x_1, x_2, \ldots, x_n)$ which minimizes $f(x)$ subject to the constraints $g_i(x) \leq 0$ for $i = 1, \ldots, m$, and $l_j(x) = 0$ for $j = 1, \ldots, p$.
The variable x is the *vector of variables*, $f(x)$ is the *objective function*, $g_i(x) \leq 0$ are the *inequality constraints* and $l_j(x) = 0$ are the *equality constraints*. The number of variables n and the number of constraints $p+m$ need not to be related. If $p+m = 0$ the problem is called an **unconstrained optimization problem**, otherwise it is called a **constrained optimization problem**. Furthermore optimization problems can be classified as linear,

quadratic, polynomial, non-linear depending upon the nature of the objective functions and the constraints. Finally depending upon the values permitted for the variables, optimization problems can be classified as integer or real valued, and deterministic or stochastic.

1.1 Lagrange Theory

Although modern mathematical programming was born towards the end of the last century, however, the history of mathematical programming dates back to the end of 1700, though limited to the case of constraints expressed by equality. In fact, optimization problems which consist in maximizing or minimizing a given function, subject to a system of constraints expressed by equality, were studied by G.L. Lagrange in the second half of the 18th century. Lagrange introduced his "multipliers" in 1778, in the fourth section of the first part of his famous book *Mécanique Analytique*,as a tool to determine the stable equilibrium configuration in a mechanical system. In *Théorie de fonctions analytiques* (1797) the multiplier method is presented throughout its generality, not referring to any specific issue of Mechanics but introduced for optimization problems.

Consider the following optimization problem with equality constraints:

$$\min f(x) \quad (\text{or} \max f(x)) \tag{1.1}$$

subject to:

$$\begin{aligned} &g_1(x) = b_1, \\ &g_2(x) = b_2, \\ &\ldots, \\ &g_m(x) = b_m. \end{aligned}$$

A solution can be found using the function:

$$L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i (b_i - g_i(x)). \tag{1.2}$$

This function is called the *Lagrangian function* of the problem. Note that (1.2) can also be written as $L(x, \lambda) = f(x) - \sum_{i=1}^{m} \lambda_i (g_i(x) - b_i)$.
The reason L is of interest is the following. Assume $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ maximizes or minimizes $f(x)$ subject to the constraints $g_i(x) = b_i$, for $i = 1, 2, \ldots, m$. Then either

(i) the vectors $\nabla g_1(x^*), \nabla g_2(x^*), \ldots, \nabla g_m(x^*)$ are linearly dependent, or

(ii) there exists a vector $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*)$ such that $\nabla L(x^*, \lambda^*) = 0$. I.e.

$$\frac{\partial L}{\partial x_1}(x^*, \lambda^*) = \frac{\partial L}{\partial x_2}(x^*, \lambda^*) = \ldots = \frac{\partial L}{\partial x_n}(x^*, \lambda^*) = 0$$

and

$$\frac{\partial L}{\partial \lambda_1}(x^*, \lambda^*) = \frac{\partial L}{\partial \lambda_2}(x^*, \lambda^*) = \ldots = \frac{\partial L}{\partial \lambda_m}(x^*, \lambda^*) = 0.$$

Of course, case (i) above cannot occur when there is only one constraint. Note that the equation

$$\frac{\partial L}{\partial \lambda_i}(x^*, \lambda^*) = 0$$

is nothing more than

$$b_i - g_i(x^*) = 0 \quad \text{or} \quad g_i(x^*) = b_i.$$

In other words, taking the partial derivatives with respect to λ does nothing more than returning the original constraints.
Once we have found candidate solutions x^*, it is not always easy to figure out whether they correspond to a minimum, a maximum or neither. The following situation is one when we can conclude. If $f(x)$ is concave and all of the $g_i(x)$ are linear functions, then any feasible x^* with a corresponding λ^* making $\nabla L(x^*, \lambda^*) = 0$ maximizes $f(x)$ subject to the constraints. Similarly, if $f(x)$ is convex and each $g_i(x)$ is linear, then any x^* with a λ^* making

$\nabla L(x^*, \lambda^*) = 0$ minimizes $f(x)$ subject to the constraints.

Now consider the following optimization problem with equality and inequality constraints:

$$\max f(x)$$

subject to:

$$\begin{aligned} &g_1(x) = b_1, \\ &\dots, \\ &g_m(x) = b_m, \\ &h_1(x) \leq d_1, \\ &\dots, \\ &h_p(x) \leq d_p. \end{aligned}$$

If you have a problem with $\geq$ constraints, convert it into $\leq$ by multiplying by -1. Also convert a minimization to a maximization. In this case the Lagrangian function is:

$$L(x, \lambda, \mu) = f(x) + \sum_{i=1}^{m} \lambda_i (b_i - g_i(x)) + \sum_{j=1}^{p} \mu_j (d_j - h_j(x)).$$

The fundamental result is the following. Assume $x^* = (x_1^*, x_2^*, \dots, x_n^*)$ maximizes $f(x)$ subject to the constraints $g_i(x) = b_i$, for $i = 1, 2, \dots, m$ and $h_j(x) \leq d_j$, for $j = 1, 2, \dots, p$. Then either

(i) the vectors $\nabla g_1(x^*), \dots, \nabla g_m(x^*), \nabla h_1(x^*), \dots, \nabla h_p(x^*)$ are linearly dependent, or

(ii) there exists a vector $\lambda^* = (\lambda_1^*, \dots, \lambda_m^*)$ and $\mu^* = (\mu_1^*, \dots, \mu_p^*)$ such that

$$\begin{aligned} &\nabla f(x^*) - \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) - \sum_{j=1}^{p} \mu_j^* \nabla h_j(x^*) = 0, \\ &\mu_j^* (h_j(x^*) - d_j) = 0 \quad \text{(Complementarity)}, \\ &\mu_j^* \geq 0. \end{aligned}$$

In general, to solve these equations, you begin with complementarity and note that either μ_j^* must be zero or $h_j(x^*) - d_j = 0$. Based on the various possibilities, you come up with one or more candidate solutions. If there is an optimal solution, then one of your candidates will be it. The above conditions are called the Kuhn-Tucker (or Karush-Kuhn-Tucker) conditions. Why do they make sense? For x^* optimal, some of the inequalities will be tight and some not. Those not tight can be ignored (and will have corresponding $\mu_j^* = 0$). Those that are tight can be treated as equalities which leads to the previous Lagrangian problem. So $\mu_j^*(h_j(x^*) - d_j) = 0$ (Complementarity) forces either the μ_j^* to be 0 or the constraint to be tight.

A special type of constraint is nonnegativity. If you have a constraint of the kind $x_k \geq 0$, you can write this as $-x_k \leq 0$ and use the above result. This constraint would get a Lagrange multiplier of its own, and would be treated just like every other constraint. An alternative is to treat nonnegativity implicitly. If x_k must be nonnegative:

1 Change the equality associated with its partial derivatives to a less than or equal to zero:

$$\frac{\partial f(x)}{\partial x_k} - \sum_i \lambda_i \frac{\partial g_i(x)}{\partial x_k} - \sum_j \mu_j \frac{\partial h_j(x)}{\partial x_k} \leq 0$$

2 Add a new complementarity constraint:

$$\left(\frac{\partial f(x)}{\partial x_k} - \sum_i \lambda_i \frac{\partial g_i(x)}{\partial x_k} - \sum_j \mu_j \frac{\partial h_j(x)}{\partial x_k}\right) x_k$$

3 Don't forget that $x_k \geq 0$ for x to be feasible.

The Karush-Kuhn-Tucker conditions give us candidate optimal solutions x^*. When are these conditions sufficient for optimality? That is, given x^* with λ^* and μ^* satisfying the KKT conditions, when can we be certain that it is an optimal solution? The most general condition available is:

1 $f(x)$ is concave

2 the feasible region forms a convex region.

While it is straightforward to determine if the objective is concave by computing its Hessian matrix, it is not so easy to tell if the feasible region is convex. A useful condition is as follows: the feasible region is convex if all of the $g_i(x)$ are linear and all of the $h_j(x)$ are convex. If this condition is satisfied, then any point that satisfies the KKT conditions gives a point that maximizes $f(x)$ subject to the constraints.

1.2 Variational Inequalities Theory

For the analysis of economic phenomena, equilibrium is a central concept. Methodologies applied to formulation, qualitative analysis, and economic equilibrium calculation include equation systems, optimization theory, complementarity theory, as well as fixed point theory. We present, in this chapter, some notes on the theory of variational inequalities; specifically, we define the problem of variational inequality that binds it to other well-known problems.

The problem of variational inequality is a general formulation of the problem that includes an overabundance of mathematical problems, including, among other things, non linear equations, complementarity problems, optimization problems and fixed point problems. Originally, variants of inequalities have been developed as a tool for studying some classes of partial differential equations such as those that appear in mechanics and have been defined over infinite-dimensional spaces.

The first variational inequality problem is the *problem with ambiguous boundary conditions* of Antonio Signorini (1959), which consists in finding the elastic balance configuration of an elastic body, which it rests on a rigid surface without friction and is only subject to its weight. This problem is known as *Signorini problem* (see [59]), name dedicated to it by his student Gaetano Fichera who resolved it in 1963.

In 1964 Guido Stampacchia, 20th-century Italian mathematician, known for his work on the theory of variational inequalities, generalized the Lax-Milgram theorem (see [60]) and coined the name "variational inequality". Further in-depth analysis of previous studies date back to 1965 by Stampacchia and Jacques-Louis Lions (see [36]).

Definition 1. (Variational Inequality Problem)
The finite-dimensional variational inequality problem, $VI(F, K)$, is to determine a vector $x^* \in K \subset \mathbb{R}^n$, such that

$$\left\langle F(x^*)^T, x - x^* \right\rangle \geq 0, \quad \forall x \in K, \tag{1.3}$$

where F is a given continuous function from K to $\mathbb{R}^n$ and K is a given closed convex set.

The specific objective function of an optimization problem is, depending on the problem, its maximization or minimization, as well as a certain set of constraints, in the case of a constrained problem. Possible objective functions include expressions that represent profits, costs, market share, portfolio risk, and so on. Potential constraints include limited budget, limits on resources, conservation equations, non-negativity constraints on variables, etc. An optimization problem typically consists of a single function goal.
Both optimization problems, constrained and not, can be formulated as problems of variational inequality. The following two propositions and theorem identify the relationship between an optimization problem and a problem of variational inequality.

Proposition 1. *Let x^* be a solution to the optimization problem:*

$$\begin{aligned} &\min f(x) \\ &\textit{subject to: } x \in K, \end{aligned} \tag{1.4}$$

where f is continuously differentiable and K is closed and convex. Then x^* is a solution to the variational inequality problem:

$$\left\langle \nabla f(x^*)^T, x - x^* \right\rangle \geq 0, \quad \forall x \in K. \tag{1.5}$$

Proof. Let $\phi(t) = f(x^* + t(x - x^*))$, for $t \in [0, 1]$. Since $\phi(t)$ achieves its minimum at $t = 0, 0 \le \phi'(0) = \nabla f(x^*)^T \cdot (x - x^*)$, that is, x^* is a solution to (1.5). □

Proposition 2. *If $f(x)$ is a convex function and x^* is a solution to $VI(\nabla f, K)$, then x^* is a solution to the optimization problem (1.5).*

Proof. Since $f(x)$ is convex,

$$f(x) \ge f(x^*) + \nabla f(x^*)^T \cdot (x - x^*), \quad \forall x \in K. \tag{1.6}$$

But $\nabla f(x^*)^T \cdot (x - x^*) \ge 0$, since x^* is a solution to $VI(\nabla f, K)$. Therefore, from (1.6) one concludes that

$$f(x) \ge f(x^*), \quad \forall x \in K,$$

that is, x^* is a minimum point of the mathematical programming problem (1.7). □

If the feaible set $K = \mathbb{R}^n$, then the unconstrained optimization problem is also a variational inequality problem. On the other hand, in the case where a certain symmetry condition holds, the variational inequality problem can be reformulated as an optimization problem. In other words, in the case when the variational inequality formulation of the equilibrium conditions underlying a specific problem is characterized by a function with a symmetric Jacobian, then the solution to the equilibrium conditions and the solution to a, particular optimization problem coincide.

Theorem 1. *Assume that $F(x)$ is continuously differentiable on K and that the Jacobian matrix*

$$\nabla F(x) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{bmatrix}$$

is symmetric and positive semidefinite. Then there is a real-valued convex function $f : K \mapsto \mathbb{R}^1$ satisfying

$$\nabla f(x) = F(x)$$

with x^* the solution of $VI(F, K)$ also being the solution of the mathematical programming problem:

$$\begin{aligned} &\min f(x) \\ &\textit{subject to: } x \in K, \end{aligned} \tag{1.7}$$

Proof. Under the symmetry assumption it follows from Green's Theorem that

$$f(x) = \int F(x)^T dx, \tag{1.8}$$

where $\int$ is a line integral. The conclusion follows from Proposition 2. □

Thus, a variational inequality problem can be reformulated as a convex optimization problem only when it maintains the symmetry condition and the positive semidefiniteness condition, although the variational inequality problem includes the optimization problem. Therefore, the variational inequality is the more general problem in that it can also handle a function $F(x)$ with an asymmetric Jacobian. Many equilibrium problems, historically, have been reformulated as optimization problems, under such a symptom hypothesis. However, the assumption in terms of applications was restrictive and precluded the more realistic modeling of multiple commodities, multiple modes and/or classes in competition. In addition, the resulting objective function was sometimes artificial, without a clear economic interpretation and simply a mathematical device.

1.3 Traffic Network Equilibrium

For users of a congested transport network to determine their travel paths from their origins to their destinations at the lowest cost is a classic network equilibrium problem. It appeared in Pigou's work in 1920: he considered a two-node transport network, two links (or paths), and was further developed by Knight in 1924. In this definition of the problem, the demand side coincides with potential travelers or consumers of the network, while the supply

side is represented by the same network, with the prices corresponding to travel costs. When the number of trips between an origin and a destination is equivalent to the travel demand given by the market price, i.e. the travel time of the trips, the equilibrium occurs.
In 1952 Wardrop stated the traffic equilibrium conditions through two principles. *First Principle*: the travel times of all the routes actually used are the same and less than those that would be experienced by a single vehicle on any unused itinerary. *Second Principle*: Average travel time is minimal.

McGuire, Beckmann and Winsten were the first in 1956 to formulate rigorously these conditions in a mathematical manner; as well as Samuelson in 1952 in the context of space price equilibrium problems where there were, however, no congestion effects. Particularly, in 1956, McGuire, Beckmann and Winsten established the equivalence between equilibrium conditions and Kuhn-Tucker's conditions of an adequately constructed optimization problem, assuming symmetry on the under lying functions. In this case, therefore, equilibrium linkage and path tracks could be obtained as a solution to a mathematical programming problem.
In 1969 Dafermos and Sparrow invented the terms "user-optimized" and "system optimized" transportation networks to distinguish between two distinct situations in which users act in a unilateral manner, in their own personal interest, to select their paths and where users choose optimal social paths, as the total costs for the system are minimized. In this latter issue, marginal costs over average costs are balanced. In the last decades a very dynamic research activity has been seen in both modeling and developing methodologies to allow the formulation and calculation of more general patterns of traffic equilibrium. General templates examples include those that allow different modes of transport or multiple user classes who perceive cost on a connection individually. In this problem domain, in fact, the theory of finite-dimensional variational inequalities has achieved its first success, beginning with Dafermos' contributions in 1980.

1.3.1 Traffic Network Equilibrium Models

This section describes a variety of traffic network equilibrium models, and provides the variational inequality formulations of the governing equilibrium conditions. Consider now a transportation network. Let a, b, c, etc., denote the links; p, q, etc., the paths. Assume that there are J Origin/Destination (O/D) pairs, with a typical O/D pair denoted by w, and n modes of transportation on the network with typical modes denoted by i, j, etc. The flow on a link a generated by mode i is denoted by f_a^i, and the user cost associated with traveling by mode i on link a is denoted by c_a^i. Group the link flows into a column vector $f \in \mathbb{R}^{nL}$, where L is the number of links in the network. Group the link costs into a row vector $c \in \mathbb{R}^{nL}$. Assume now that the user cost on a link and a particular mode may, in general, depend upon the flows of every mode on every link in the network, that is,

$$c = c(f) \tag{1.9}$$

where c is a known smooth function. These cost functions contain the linear, separable cost function as a special case. The travel demand of potential users of mode i traveling between O/D pair w is denoted by d_w^i and the travel disutility associated with traveling between this O/D pair using the mode is denoted by λ_w^i. Group the demands into a vector $d \in \mathbb{R}^{nJ}$ and the travel disutilities into a vector $\lambda \in \mathbb{R}^{nJ}$. The flow on path p due to mode i is denoted by x_p^i. Group the path flows into a column vector $x \in \mathbb{R}^{nQ}$, where Q denotes the number of paths in the network. The conservation of flows equations are as follows. The demand for a mode and O/D pair must be equal to the sum of the flows of the mode on the paths joining the O/D pair, that is,

$$d_w^i = \sum_{p \in P_w} x_p^i, \quad \forall i, w \tag{1.10}$$

where P_w denotes the set of paths connecting w. A nonnegative path flow vector x which satisfies (1.10) is termed feasible. Inoltre, il flusso su un arco è dato dalla somma dei flussi sui percorsi che contengono quell'arco, ossia:

$$f_a^i = \sum_p x_p^i \delta_{ap} \tag{1.11}$$

where $\delta_{ap} = 1$ if $a \in p$, 0 altrimenti. A user traveling on path p using mode i incurs a user (or personal) travel cost C_p^i satisfying

$$C_p^i = \sum_a c_a^i \delta_{ap} \tag{1.12}$$

in other words, the cost on a path p due to mode i is equal to the sum of the link costs of links comprising that path and using that mode. The traffic network equilibrium conditions are given below.

Definition 2. *A flow pattern (f^*, d^*) compatible with (1.10) and (1.11) is an equilibrium pattern if, once established, no user has any incentive to alter his/her travel arrangements. This state is characterized by the following equilibrium conditions, which must hold for every mode i, every O/D pair w, and every path $p \in P_w$:*

$$C_p^i \begin{cases} = \lambda_w^i & \text{if } x_p^{i*} > 0, \\ \geq \lambda_w^i & \text{if } x_p^{i*} = 0. \end{cases} \tag{1.13}$$

where λ_w^i is the equilibrium travel disutility associated with the O/D pair w and mode i.

1.3.1.1 The Elastic Demand Model with Disutility Functions

In this section assume that there exist travel disutility functions, such that

$$\lambda = \lambda(d), \tag{1.14}$$

where λ is a known smooth function. That is, let the travel disutility associated with a mode and an O/D pair depend, in general, upon the entire demand pattern. Let $\mathcal{K}$ denote the feasible set defined by:

$$\mathcal{K} = \{(f, d) | \quad \exists x \geq 0 | \quad (1.10), (1.11) \text{ hold}\}. \tag{1.15}$$

The variational inequality formulation of equilibrium conditions (1.13) is given in the next theorem. Assume that λ is a row vector and d is a column vector.

Theorem 2. *A pair of vectors $(f^*, d^*) \in \mathcal{K}$ is an equilibrium pattern if and only if it satisfies the variational inequality problem*

$$c(f^*) \cdot (f - f^*) - \lambda(d^*) \cdot (d - d^*) \geq 0, \quad \forall (f, d) \in \mathcal{K}. \tag{1.16}$$

Observe that in the above model the feasible set is not compact. Therefore, a condition such as strong monotonicity would guarantee both existence and uniqueness of the equilibrium pattern (f^*, d^*); in other words, if one has that

$$\begin{aligned} &[c(f^1) - c(f^2)] \cdot [f^1 - f^2] - [\lambda(d^1) - \lambda(d^2)] \cdot [d^1 - d^2] \\ &\geq \alpha(||f^1 - f^2||^2 - ||d^1 - d^2||^2), \quad \forall (f^1, d^1), (f^2, d^2) \in \mathcal{K}, \end{aligned} \tag{1.17}$$

where $\alpha > 0$ is a constant, then there is only one equilibrium pattern. Condition (1.18) implies that the user cost function on a link due to a particular mode should depend primarily upon the flow of that mode on that link; similarly, the travel disutility associated with a mode and an O/D pair should depend primarily on that mode and that O/D pair. The link cost functions should be monotonically increasing functions of the flow and the travel disutility functions monotonically decreasing functions of the demand.

1.3.1.2 The Elastic Demand Model with Demand Functions

In this section assume that there exist travel demand functions, such that

$$d = d(\lambda) \tag{1.18}$$

where d is a known smooth function. Assume here that d is a row vector. In this case, the variational inequality formulation of equilibrium conditions (1.13) is given in the subsequent theorem, whose proof appears in Dafermos and Nagurney (1984a).

Theorem 3. *Let $\mathcal{M}$ denote the feasible set defined by*

$$\mathcal{M} = \{(f, d, \lambda) | \lambda \geq 0, \exists x \geq 0 | (1.10), (1.11) \ \textit{hold}\}. \qquad (1.19)$$

The vector $X^* = (f^*, d^*, \lambda^*) \in \mathcal{M}$ is an equilibrium pattern if and only if it satisfies the variational inequality problem:

$$F(X^*) \cdot (X - X^*) \geq 0, \quad \forall X \in \mathcal{M}, \qquad (1.20)$$

where $F : \mathcal{M} \mapsto \mathbb{R}^{n(L+2J)}$ is the function defined by

$$F(f, d, \lambda) = (c(f), -\lambda^T, d - d(\lambda)). \qquad (1.21)$$

1.3.1.3 The Fixed Demand Model

For completeness, the fixed demand model is presented is this section. Specifically, it is assumed that the demand d_w^i is now fixed and known for all modes i and all origin/destination pairs w. In this case, the feasible set $\mathcal{K}$ would be defined by

$$\mathcal{K} = \{f | \quad \exists x \geq 0 | \quad (1.10), (1.11) \text{ hold}\}. \qquad (1.22)$$

It is easy to verify (see also Smith (1979) and Dafermos (1980)) that the variational inequality governing equilibrium conditions (1.13) for this model would be given as in the subsequent theorem.

Theorem 4. *A vector $f^* \in \mathcal{K}$, is an equilibrium pattern if and only if it satisfies the variational inequality problem*

$$c(f^*) \cdot (f - f^*) \geq 0, \quad \forall f \in \mathcal{K}. \qquad (1.23)$$

Chapter 2

Some transplant models in the literature

In this chapter, after a brief introduction about the birth of transplants and the different research fields, we present some mathematical models related to the transplant process existing in the literature.

2.1 Organ transplants: birth and studies

The popular fantasy has always been fascinated by the ability to improve health or lengthen a person's life through the transplantation of diseased organs with other healthy individuals taken from individuals of the same breed or even different breed. Already in mythology, in legends and art representations this idea was present. Medical saints Cosmas and Damian, according to an ancient tradition, were the fathers of transplant: they replaced the gangrenous leg of their sacristan by the healthy leg of a newly deceased Ethiopian man. In 1902, Alexis Carrel, made the first crucial step for organ transplantation, adopting for the first time a technique for joining blood vessels, thus beginning the scientific history of transplants.
In the 1940, the next step was when Peter Medawar discovered the basis of "compatibility", attempting the burned skin transplant on the victims

severely burned by the bombings in London during the second World War. But the first real transplant was carried out in 1954 in Boston (US) by surgeon Murray which completed the kidney transplant between a living donor and a consanguineous and genetically identical recipient.
Several research studies confirm the importance of the transplantation theme, which is evidenced by numerous works in medical literature but also in mathematical literature. In this latter area some topics are treated, such as: optimization of times, fundamental in the transplant process; the best allocation of organs to transplant centers; the management of waiting lists; cost minimization of each stage of the transplant process. In the next paragraphs we will briefly treat some of these models.

2.2 Optimizing the supply chain design for organ transplants

Considering the crucial role of organ transplant time in the transplant process, in [4] the authors have studied a model to minimize it. The model, applied to organ transplants in Belgium, is developed as a mixed integer programming problem (MIP).
When a potential donor is known in a hospital, the Belgian transplant process begins. The hospital of donor works with one of the eight Transplant Centers in Belgium and informs the center transplant coordinator, who is responsible for collecting blood samples at the potential donor hospital: these ones will be transported to the transplant center laboratory. At the same time, the shipping agent, who is responsible for all the phases of transporting of the organ to be transplanted, returns to its base.
All data are sent to Eurotrasplant whose database contains the list of all potential recipients waiting for donor organs. Therefore, Eurotransplant informs of the availability of organs the coordinator of the recipient's transplant center; if the coordinator agrees, the recipient is notified, while the

destination of organ donation is notified to the donor hospital transplant coordinator. The donor organ is withdrawn by the donor hospital transplant coordinator team. Then the organ is transported and implanted at the transplant recipient center. At the end of the process, all the actors go back to their locations.

Let $c, c', r, r' \in C$ be the transplant centers, $o \in O$ the organs to be transplanted, $v \in V$ the shipping agents, $h, h' \in H$ the hospitals, $m \in M$ the municipalities and $a \in A$ the airports. Infact it is also possible that the donor organ comes from abroad (by plane). In this case, the organ is transported from the airport to the transplant center of the recipient.
Let D^d_{oc} be the recipient demand for transplantations of organ o at center c coming from domestic donors, D^a_{oc} the recipient demand for transplantations of organ o at center c coming from donors abroad, S^d_{oh} the donor supply of organ o at hospital h for domestic recipients, S^a_{oh} the donor supply of organ o at hospital h for recipients abroad, k the fixed costs for installing a transplant center, k^o the cost of installing a center capable of transplanting organ o, B the available budget, W the number of required shipping agents, t^{isch} the maximal cold ischemia time for organ o (time interval in which the organ remains in good conditions), p_{rc} the r^{th} closest center to center $c, r \in \{1, 2, \ldots, |C| - 1\}$, T_o the maximal allowed traveling time between any municipality and an installed center for organ o, $t^{sh \to ho}_{vh}$ the traveling time from shipping agent v to hospital h, $t^{mu \to ce}_{mc}$ the traveling time from municipality m to transplant center c, $t^{ho \to ce}_{hc}$ and $t^{ce \to ho}_{ch}$ the traveling time from hospital h to transplant center c and viceversa, and $t^{ai \to ce}_{ac}$ the traveling time from airport a to transplant center c.
The decision variables are: $y_{oc} \in \{0, 1\}$ that is 1 if center c is installed for organ o and 0 otherwise; $z_c \in \{0, 1\}$ that is 1 if center c is installed for at least one organ o; $w_v \in \{0, 1\}$ that is 1 if shipping agent v is selected and 0 otherwise. Furthermore let $x^{ho \to re}_{ohc}$ be the flow of organ o from hospital h to receiver center c, x^{over}_{ohc} extra flow of organ o from hospital h to receiver center

c because of oversupply at hospital h, $x_{ohc}^{ho\to do}$ flow of organ o from hospital h to donor center c, $x_{oac}^{ai\to re}$ flow of organ o from airport a to receiver center c, $x_{vh}^{sh\to ho}$ flow from shipping agent v to hospital h, d_{oc}^{a} total recipient demand for transplantations of organ o at center c coming from donors abroad, $d_{oc'c}^{d,shift}$ recipient demand for transplantations of organ o coming from domestic donors shifted from center c' to c, $d_{oc'c}^{a,shift}$ recipient demand for transplantations of organ o coming from abroad shifted from center c' to c.
Therefore the model is:

$$\min\Bigg\{\sum_{v\in V}\sum_{h\in H} t_{vh}^{sh\to ho}x_{vh}^{sh\to ho}+\sum_{o\in O}\sum_{h\in H}\sum_{c\in C}(t_{hc}^{ho\to ce}+t_{ch}^{ce\to ho})x_{ohc}^{ho\to do} + \sum_{o\in O}\sum_{h\in H}\sum_{c\in C} t_{hc}^{ho\to ce}x_{ohc}^{ho\to re}+\sum_{o\in O}\sum_{a\in A}\sum_{c\in C} t_{ac}^{ai\to ce}x_{oac}^{ai\to re}\Bigg\} \tag{2.1}$$

subject to the constraints:

$$y_{oc}\le z_c \quad \forall o\in O, c\in C; \tag{2.2}$$

$$\sum_{c\in C}\Big(kz_c+\sum_{o\in O}k_o y_{oc}\Big)\le B; \tag{2.3}$$

$$\sum_{v\in V} w_v = W; \tag{2.4}$$

$$x_{vh}^{sh\to ho}\le\Big(\sum_{o\in O}(S_{oh}^{d}+S_{oh}^{a})\Big)w_v \quad \forall v\in V, h\in H; \tag{2.5}$$

$$\sum_{v\in V} x_{vh}^{sh\to ho}=\sum_{o\in O}(S_{oh}^{d}+S_{oh}^{a}) \quad \forall h\in H; \tag{2.6}$$

$$\sum_{h\in H}\sum_{c\in C} x_{ohc}^{ho\to re}=\sum_{c\in C} D_{oc}^{d} \quad \forall o\in O; \tag{2.7}$$

$$x_{ohc}^{ho\to re}=0 \quad \forall o\in O, h\in H, c\in C|t_{hc}^{ho\to ce}>t_o^{isch}; \tag{2.8}$$

$$x_{ohc}^{ho\to re}=\Big(\frac{D_{oc}^{d}y_{oc}+\sum_{c'\in C} d_{oc'c}^{d,shift}}{\sum_{c'\in C} D_{oc'}^{d}}\Big)S_{oh}^{d}+x_{ohc}^{over} \quad \forall o\in O, h\in H, c\in C|t_{hc}^{ho\to ce}\le t_o^{isch}; \tag{2.9}$$

$$d^{d,shift}_{oc'c} \geq (y_{oc} - y_{oc'} - \sum_{r'=1}^{r-1} y_{op_{r'c'}}) D^d_{oc'}$$
$$\forall o \in O, c \in C, c' \in C, r \in \{1, 2, \ldots, |C|-1\} | p_{rc'} = c; \quad (2.10)$$

$$\sum_{c \in C} x^{ho \to do}_{ohc} = S^d_{oh} + S^a_{oh} \quad \forall o \in O, h \in H; \quad (2.11)$$

$$\sum_{c \in C} x^{ho \to re}_{ohc} = S^d_{oh} \quad \forall o \in O, h \in H; \quad (2.12)$$

$$x^{ho \to re}_{ohc} \leq S^d_{oh} y_{oc} \quad \forall o \in O, h \in H, c \in C; \quad (2.13)$$

$$x^{ho \to do}_{ohc} \leq (S^d_{oh} + S^a_{oh}) y_{oc} \quad \forall o \in O, h \in H, c \in C; \quad (2.14)$$

$$x^{over}_{ohc} \leq S^d_{oh} y_{oc} \quad \forall o \in O, h \in H, c \in C; \quad (2.15)$$

$$d^a_{oc} = D^a_{oc} y_{oc} + \sum_{c' \in C} d^{a,shift}_{oc'c} \quad \forall o \in O, c \in C; \quad (2.16)$$

$$d^{a,shift}_{oc'c} \geq (y_{oc} - y_{oc'} - \sum_{r'=1}^{r-1} y_{op_{r'c'}}) D^a_{oc'}$$
$$\forall o \in O, c \in C, c' \in C, r \in \{1, 2, \ldots, |C|-1\} | p_{rc'} = c; \quad (2.17)$$

$$d^a_{oc} = \sum_{a \in A} x^{ai \to re}_{oac} \quad \forall o \in O, c \in C; \quad (2.18)$$

$$\sum_{c \in C | t^{mu \to ce}_{mc} \leq T_o} y_{oc} \geq 1 \quad \forall o \in O, m \in M; \quad (2.19)$$

$$z_c \in \{0, 1\} \quad \forall c \in C; \quad (2.20)$$

$$y_{oc} \in \{0, 1\} \quad \forall o \in O, c \in C; \quad (2.21)$$

$$w_v \in \{0, 1\} \quad \forall v \in V; \quad (2.22)$$

$$x^{ho \to re}_{ohc} \geq 0 \quad x^{over}_{ohc} \geq 0 \quad x^{ho \to do}_{ohc} \geq 0 \quad \forall o \in O, h \in H, c \in C; \quad (2.23)$$

$$x^{ai \to re}_{oac} \geq 0 \quad \forall o \in O, a \in A, c \in C; \quad (2.24)$$

$$x^{sh \to ho}_{vh} \geq 0 \quad \forall v \in V, h \in H; \quad (2.25)$$

$$d^a_{oc} \geq 0 \quad \forall o \in O, c \in C; \quad (2.26)$$

$$d^{d,shift}_{oc'c} \geq 0 \quad d^{a,shift}_{oc'c} \geq 0 \quad \forall o \in O, c \in C, c' \in C. \quad (2.27)$$

The objective function (2.1) minimizes the total transportation time. The first term indicates the traveling time from shipping agent to donor hospital. The second term indicates the traveling time from donor hospital to donor transplant center and back. The third term contains the traveling time from donor hospital to recipient transplant center. The fourth term indicates the traveling time between the airports and recipient transplant centers for organs that come from abroad.

Constraint set (3.2) ensures that a center can only be installed for a particular organ if the center is opened. Constraint (3.3) defines the budget restriction. Constraint (3.4) ensures the correct number of selected shipping agents. Constraint set (3.5) guarantees that there is only flow from a shipping agent to a hospital if that shipping agent is selected. Observe that $\sum_{o \in O}(S^d_{oh} + S^a_{oh})$ is the maximal flow between hospital and any shipping agent, and hence this amount is a kind of big M value. Constraint set (3.6) ensures that all organs that become available at a particular hospital are served by a shipping agent. Constraint set (3.7) makes sure that the total demand for organs at the recipient transplant centers coming from domestic donors equals the total flow out of the donor hospitals. Constraint sets (3.10) and (3.9) calculate a particular organ flow from a donor hospital to a recipient transplant center. This flow equals 0 if the travel time between the hospital and the center is larger than the maximal cold ischemia time. Otherwise, this flow consists of two parts. The first part is a fraction of the hospital?s supply ($= S^d_{oh}$). This fraction equals the demand of the center increased by the demand from (nearby) closed centers shifted to this center, determined by constraint set (2.10), divided by the total demand over all centers. The second part ($= x^{over}_{ohc}$) can only be positive, since it refers to a situation where there is a surplus supply at the hospital (i.e., when the total demand from all recipient transplant centers within a traveling time smaller or equal to the maximal cold ischemia time is insufficient to cover the total supply of the donor hospital).

Constraint set (2.10) calculates the shifted demand from each closed center to each opened center . Obviously, the demand from $c'(= D^d_{oc'})$ can only be shifted from c' to c if c' is closed and c is open, a situation which is described by the first two terms, $y_{oc} - y_{oc'}$, in the right hand side. Furthermore, since demand is only shifted to the closest open center, the shifted demand can again be zero if there is any center open closer to center c' than center c, a factor which explains the last term, $-\sum_{r'=1}^{r-1} y_{op_{r'c'}}$. Constraint set (2.11) ensures that the total flow out of a donor hospital to the donor transplant centers equals the total organ supply (for domestic transplantations and transplantations abroad) at that hospital. Recall that each organ, whether it will be transplanted by a domestic center or a center abroad, generates a flow from the donor hospital to the donor transplant center (blood samples) and back (transplant coordinator and transplant team). Constraint set (2.13) ensures that all supply at each donor hospital is sent out to the recipient transplant centers. Constraint sets (2.13) and (2.14) guarantee that there is no flow to a closed center, that is to say, neither to a recipient transplant center, see (2.13), nor to a donor transplant center, see (2.14). In these constraints, S^d_{oh} and $(S^d_{oh} + S^a_{oh})$ acts as big M. Constraint set (2.15) ensures that there can only exist an extra flow of an organ from a hospital to a receiver center because of oversupply at that hospital, if this receiver center is opened. Constraint set (2.16) calculates, for each organ, the number of transplantations in a particular center from donors abroad. This number equals the center's international demand, increased by the shifted international demand of (nearby) closed centers. The latter is determined by constraint set (2.17), which is similar to the shifted domestic demand calculated in constraint set (2.10). Constraint set (2.18) ensures that this international demand results in a flow from a chosen airport to the center. Note that, in order to minimize the objective function, the model will always select the closest airport, which is conform to reality, as international organs enter the country through the airport closest to the transplant center of the recipient. Constraint set (2.19) models the covering restriction: for each organ, at least one transplant center must be installed within a particular travel time of each municipality in the country. Finally, constraint sets (2.20) to (2.27) define the decision variables.

2.3 A new organ transplantation location–allocation policy: a case study of Italy

The localization model for the optimal organization of the transplant system developed in [8] by the authors, uses an approach based on a mathematical programming formulation. The authors focused on the critical role played by time in the transplantation process and on the territorial distribution of transplantation centers. The aim of this document is to allocate transplantable organs between regions, seeking to achieve maximum regional equity in health care.
In the USA there are some organizations in the transplant system called OPO (Organ Procurement Organization) which play a fundamental role in the design of a fair end equitable transplant system. The aforementioned organizations are responsible for procuring all organs donated in their region and allocating them to the candidates registered on their transplant waiting lists, operating in geographically distinct regions. Each of them may serve none, one or more than one transplant center, considering that each OPO has a single main transplant center. This organization leads to an equivalence between the location of OPOs and the location of the Transplant Centers.

Let $I = \{i : i = 1, 2, \ldots, M\}$ represent the set of explantation centers and $J = \{j : j = 1, 2, \ldots, N\}$ be the set of potential transplant centers. Each explantation center $i \in I$ will belongs to one OPO $j \in J$ and will refer to one transplant center. We fix the number p of OPOs to be opened. We assume potential recipients of explanted organs (demand points) aggregated to P locations; let $L = \{l : l = 1, 2, \ldots, P\}$ be the set of potential recipients with the associated annual demand h_l with $l \in L$. Once an organ is available and a clinical allocation policy assigns the organ to the first ranked waiting host, this has to travel to the transplant center. The distance the patient travel to reach the transplant facility is crucial. The organ quality and the likelihood of a successful transplant decrease as the elapsed time from organ retrieval to organ transplantation increases. A measure of travel time between the patient $l \in L$ and the transplant facility $j \in J$ is measured by means of terrestrial distance d_{lj}. The distance traveled by the organ from the donor hospital $i \in I$ to the transplant center $j \in J$ is the aerial distance a_{ij}. This is motivated by the fact that transportation of explanted organ is usually performed with the hospital emergency helicopter.

The mathematical programming formulation for the location model (TRALOC: Tansplant Location Allocation Model) is:

$$\min \left\{ \sum_{i=1}^{M} \sum_{j=1}^{N} a_{ij} x_{ij} + \sum_{l=1}^{P} \sum_{j \in T_l} h_l d_{lj} y_{lj} + E \right\} \tag{2.28}$$

subject to the constraints:

$$\sum_{j=1}^{N} x_{ij} = 1, \quad i = 1, \ldots, M; \tag{2.29}$$

$$\sum_{j \in T_l} y_{lj} = 1, \quad l = 1, \ldots, P; \tag{2.30}$$

$$y_{lj} \leq z_j, \quad l = 1, \ldots, P, \quad j = 1, \ldots, N; \tag{2.31}$$

$$x_{ij} \leq z_j, \quad i = 1, \ldots, M, \quad j = 1, \ldots, N; \tag{2.32}$$

$$\sum_{j=1}^{N} z_j = p; \tag{2.33}$$

$$E \geq \sum_{l=1}^{P} h_l y_{lj}, \quad j = 1, \ldots, N; \tag{2.34}$$

$$x_{ij} \in \{0, 1\}, \quad i = 1, \ldots, M, \quad j = 1, \ldots, N; \tag{2.35}$$

$$y_{lj} \in \{0, 1\}, \quad l = 1, \ldots, P, \quad j = 1, \ldots, N; \tag{2.36}$$

$$z_j \in \{0, 1\}, \quad j = 1, \ldots, N; \tag{2.37}$$

Here $T_l = \{j \in J | d_{lj} \leq r\}$, where r is a coverage distance related to the cold-ischemia time t. Note that T_l is the set of all those candidates that are within an acceptable distance of the transplant center j. The maximum value preset for the travel time is the cold-ischemia time. Because the cold-ischemia time is different for different organs, this data cannot be aggregated. This implies that we have to solve different problems depending on the organ we are considering. The binary decision variables are x_{ij}, y_{lj}, z_j. If the donor center i belongs to the OPO j this implies $x_{ij} = 1$. In particular $y_{lj} = 1$ if the demand point l belongs to the OPO/transplant center j. The number of OPO to be located ($z_j = 1$ if OPO j is activated) is fixed to p (see constraint (2.33)). Constraint sets (2.29) and (2.30) require that each demand node, as well as each donor center, has to be assigned to exactly one OPO. Constraint sets (2.31) and (2.32) restrict these assignments only to open OPOs.
The objective function, to be minimized, is composed by three terms. The first term $\sum_{i=1}^{M} \sum_{j=1}^{N} a_{ij} x_{ij}$ minimizes the total distance between explantation-performing centers and transplant centers. This term reflects the first component of the waiting time. The second term $\sum_{l=1}^{P} \sum_{j \in T_l} h_l d_{lj} y_{lj}$ minimizes the demand weighted total distance traveled by the patients to the transplant center in order to receive an explanted organ. The third part of the objective minimizes the maximum size of waiting list given that a patient belonging to the OPO j is also part of the OPO's waiting list. In fact the decision variable E is the maximum over j of the sum of requests arising from OPO j (see constraint (2.34)). Since the maximum waiting list size across OPOs is to be minimized, we want to find a set of locations that will give the smallest maximum waiting list size possible when evaluated for all OPOs.

2.4 Redesigning Organ Allocation Boundaries for Liver Transplantation in the United States

To construct alternative liver allocation boundaries that achieve more geographic equity in access to transplants than the current system, the authors in [32] apply mathematical programming. In the United States, in fact, the alleged causes of disparities is the administrative boundaries of organ allocation that limit the sharing of organs between the regions. Optimum boundary performance were evaluated and compared to that of current allocation system using discrete event simulation. The mathematical programming approach has the dual objective of identifying optimal positions for liver transplant centers and identifying new boundaries for Organ Procurement Organizations replacing the existing boundaries of the OPO, which are predominantly determined by political issues.

The model addresses the clustering problem of a set of transplant centers selected for activation into a predefined number of clusters. Each of these clusters represents an OPO. The resulting OPOs are determined so that they are balanced both in terms of organ supply / demand ratio and in terms of total number of transplant centers belonging to the OPO. Each of OPO boundary is determined by the union of the service areas associated with the transplant centers belonging to the OPO. Therefore, an important constraint to consider when defining the cluster is the service areas contiguity. To achieve the target, the model assumes a graph $G = (V, E)$ as an input, where each vertex $i \in V$ is associated with a transplant center and there exists an arc $(i, j) \in E$ between vertex i and j if the corresponding service areas have a common boundary. Each vertex i of this graph is associated with two weights: w_i and h_i represent respectively the total supply and the total demand associated with the transplant center represented by the vertex. A super vertex s is added to the graph and is connected to each vertex of the graph with the set of arcs (s, i), $\forall i \in V$. Then the resulting graph is such that the total number of vertices is equal to $p + 1$, where p is a fixed number of transplant centers to open between a set of possible candidates. The model seeks a spanning tree T_s of G rooted in s such that the total number of root children is equal to the total number of clusters to be defined. In this way, the vertices of each T_i rooted at vertex i, i.e. one of the supervertex children, represent the whole of the transplant centers belonging to the cluster. The subtree connection ensures contiguity of the service area associated with the cluster. In addition, each subtree is such that the ratio of the sum of the weights w_i associated with the vertices of the subtree and the sum of weights h_i associated with the vertices of the subtree is greater than or equal to a predetermined threshold α. The objective function of the model is to minimize the maximum number of vertices in each of the resulting subtrees, ensuring that the resulting clusters are also balanced in terms of the total number of belonging transplantation centers.

Let $O = 1, 2, \ldots, l$ be the index set of the clusters that need to be defined. Then let y_{ik} be a binary variable that is equal to 1 if vertex $i \in V$ belongs to cluster $k \in O$ and is equal to 0 otherwise; x_{ijk} a binary variable that is equal to 1 if arc $(i, j) \in E$, that connects vertices i and j in the cluster k, is selected to be in the spanning tree and is equal to 0 otherwise; u_i defined on each vertex $i \in V$, assigns a label to each vertex of the graph. In particular, such a labeling ensures any directed arc that belongs to the optimum spanning tree goes from a vertex with a lower label to a vertex with a higher label. Hence, variables y_{ik} are used to define the clusters, while variables u_i and x_{ijk} are used to define the final spanning tree.

The mathematical formulation is:

$$\min \max \left(\sum_{i \in V} y_{ik} \right) \tag{2.38}$$

subject to the constraints:

$$\sum_{(s,j)\in E} x_{sjk} = 1, \quad \forall k \in O; \tag{2.39}$$

$$\sum_{k\in O}\sum_{(i,j)\in E} x_{ijk} = 1, \quad \forall j \in V, j \neq s; \tag{2.40}$$

$$\sum_{k\in O} x_{ijk} \leq 1, \quad \forall (i,j) \in E; \tag{2.41}$$

$$x_{ijk} \leq y_{ik}, \quad \forall (i,j) \in E, i \neq s, \forall k \in O; \tag{2.42}$$

$$y_{ik} \leq \sum_{(i,j)\in E} x_{ijk}, \quad \forall i \in V, i \neq s, \forall k \in O; \tag{2.43}$$

$$u_s = 0; \tag{2.44}$$

$$1 \leq u_i \leq p, \quad \forall i \in V, i \neq s; \tag{2.45}$$

$$(p+1)x_{ijk} + u_i - u_j + (p-1)x_{jik} \leq p, \quad \forall (i,j) \in E, i \neq s, \forall k \in O; \tag{2.46}$$

$$\sum_{k\in O} y_{ik} = 1, \quad \forall i \in V, i \neq s; \tag{2.47}$$

$$\sum_{i\in V, i\neq s} w_i y_{ik} \leq \alpha \sum_{i\in V, i\neq s} h_i y_{ik}, \quad \forall k \in O; \tag{2.48}$$

$$\sum_{i\in V, i\neq s} y_{ik} \geq 1, \quad \forall k \in O; \tag{2.49}$$

The objective function (2.38) minimizes the maximum cardinality of the resulting clusters. Constraints (2.39) ensure that the total number of children of the root s is equal to the total number of clusters that need to be defined. Constraints (2.40) ensure that each vertex has exactly one entering arc. Each arc can be associated with at most one cluster, which is ensured by constraints (2.41). Constraints (2.42) and (2.43) are logical constraints linking the binary variables. The spanning tree is defined by the classical MTZ constraints (2.44, 2.45, 2.46). Constraints (2.47) ensure that each vertex belongs exactly to one cluster. The structure of the cluster is defined by constraints (2.48) and (2.49). In particular, each cluster must not be empty (constraints (2.49)) and total supply/demand ratio at each cluster must be greater than or equal to a predefined threshold α (constraints (2.48)).

Chapter 3

A Network Model for Minimizing the Total Organ Transplant Costs

Nowadays, many organs such as kidney, liver, pancreas, intestine, heart, as well as lungs, can be safely transplanted. Sometimes organ transplantation is the only possible therapy, for instance for patients with end-stage liver diseases, and the preferred treatment, for instance for patients with end-stage renal diseases. As a consequence, the demand of organs has greatly exceeded the offer and has become a key tool to cure diseases. In many countries the costs to receive an organ, which are often very expensive, are all charged by the National Health Service. In our paper, we aim at presenting a mathematical model, based on networks, which allows us to minimize the total costs associated with organ transplants. We find the related optimality conditions and the variational inequality formulation. Some existence and uniqueness results as well as the Lagrange formulation are stated and some numerical examples are studied.

3.1 Introduction

The first kidney transplant was successfully performed in 1954 by Joseph Murray in Boston. Starting from this surgery, the organ transplantations have become the most important therapy for many diseases (see [1]). The improvement of surgical techniques and medicines has reduced the number of organ transplants, but still the demand far exceeds the amount of donated organs and therefore waiting lists are very long. Furthermore, it is worth remarking that not all the donated organs are compatible with the body of a recipient. Several studies have been done on the management of waiting lists (see [7], [19], [55]).

Moreover, once the organ is extracted from the donor's body, it remains in good conditions only for a short time interval, called *cold ischemia time*. So the transport times cannot be neglected, as well as the allocation of the hospitals and of the transplant centers. In the literature you can find some papers on the time minimization and on the optimal allocation of transplant centers (see, for instance, [4], [8], [69]).

The existing literature on organ transplant problems is very rich. In [4] Beliën, De Boeck, Colpaert, Devesse, and Van den Bosschehas develop a model for the minimization of the transport time, since it plays a crucial role in the transplant process. The model is studied as a mixed integer programming problem (MIP) and is applied to organ transplants in Belgium. In [8] Bruni, Conforti, Sicilia, and Trotta present a location model for the optimal organization of the transplant system. The approach is based on a mathematical programming formulation. The authors focus both on the critical role of time in transplantation process as well as on a spatial distribution of transplant centers. Thir aim is the allocation of transplantable organs across regions with the objective of attaining regional equity in health care. In [1] Alagoz, Schaefer, and Roberts describe a mixed integer linear programming (MILP) long-term decision model to optimize the location of organ transplant centers. Their objective is to minimize the sum of the weighted time components between the moment a donor organ becomes available and its transplantation into the recipient's body. The model is applied to the Belgian organ transplant path. In [56] the authors study the optimization of the effectiveness of organ allocations. The mathematical model simulates and analyzes the Organ Procurement and Transplantation Network, which is a system for organ distribution established by the National Organ Transplant Act. It analyzes the organ matching process and its important factors including compatibility, region, age, urgency of patient, and waitlist time. In [?] the authors apply a mathematical programming model to construct alternative liver allocation boundaries that achieve more geographic equity in access to transplants than the current system. In [33] the authors consider the problem of maximizing the efficiency of intra-regional transplants through the design of liver harvesting regions. Cadaveric liver transplantation is hindered in the United States by donor scarcity and rapid viability decay. Given these difficulties, the current U.S. liver transplantation and allocation policy attempts to balance allocation likelihood and geographic proximity by allocating cadaveric livers hierarchically. They formulate the problem as a set partitioning problem that clusters organ procurement organizations (OPOs) into regions and formulate the pricing problem as a mixed-integer program. Since the optimal solution depends on the initial design of geographic decomposition, they develop an iterative procedure that integrates branch and price with local search to alleviate this dependency.

Another important feature of organ transplantations is represented by the costs associated with each transplant, including hospital and surgery costs, medical teams and organs transportation costs, and disposal costs.

In this paper we focus our attention on such costs and present a network model for the minimization of the total costs associated to organ transplant.

The underlying network structure constituted by three levels: the first tier of nodes is represented by the transplant centers from which the medical teams reach the donor hospitals (the second tier of nodes) where they perform the organ explant and then the medical teams and the organs go back to the transplant centers (the third tier of nodes) where the organ transplant is performed.

In Italy the transplant system is the result of a complex organizational path that begins with the identification of the potential donor and ends with the surgery. It represents the conclusion of a long process that involves experts from different disciplines. Therefore it is important to have an efficient organizational model which can meet the different needs. The first organizational structure dedicated to the development and coordination of donations procedures associated with them dates back to 1976 with the foundation of the *Nord Italia Transplant program (NITp)*. Afterwards other interregional structures were created in the national territory: in 1987 the *Coordinamento Centro Sud Trapianti (CCST)* was founded and in 1989 the *Associazione Interregionale Trapianti (AIRT)* and the *Sud Italia Transplant (SIT)*. Finally in 1998 the majority of the regions that had joined the CCST and SIT merged into one organization, the *Organizzazione Centro Sud Trapianti (OCST)* (see Fig. 3.1).

Figure 3.1: Italian System

Such organizations, whose tasks are different, have the common aim of responding adequately to the needs of patients in waiting lists. Two organizations have then joined this organizational system: in 1994 the *Consulta Tecnica Permanente* and in 1999 the *Centro Nazionale Trapianti*. The latter, in particular, reviewed the existing organizational network, representing one of the most important novelties of the national transplant system. Recently, in 2014, the system has undergone a metamorphosis (see Fig. 3.4): the regions were merged into two macro interregional areas (north-central and south-central areas).

Figure 3.2: Italian System

In Italy the network that manages the explant and transplant activities consisted of four levels.

- *Local coordination level*. The local coordination immediately reports the donor information to the regional or interregional center and to the National Transplant Center, for the organs allocation; it establishes relationships with the donors' families, organizes the activities for the information, the education and the cultural growth of the population in the field of transplantation. The local coordinator's task consists in implementing strategies for the identification of all potential donors.

- *Regional coordination level.* The regional coordination has the task of collecting and transmitting the data of the patients waiting for a transplant, the removal activity and the relationships with the reanimation department. The regional center employs one or more immunology laboratories where the immunological tests are performed. It takes care of the relations with the interregional center, with the regional health authorities and with the voluntary associations.

- *Interregional coordination level.* In the past it consisted of three organizations having relationships with the regional centers in order to report the possible donors and to allocate the excesses of organs; also, they managed emergencies, relationships with the other interregional centers and with the Centro Nazionale Trapianti. These organizations were: *AIRT*, consisting of the regions Valle d'Aosta, Piemonte, Emilia Romagna, Tuscany, Puglia and the autonomous province of Bolzano; *NITp*, consisting of the regions Lombardia, Liguria, Friuli Venezia Giulia, Veneto, Marche and the autonomous province of Trento; *OCST*, consisting of the regions Abruzzo, Molise, Umbria, Lazio, Campania, Basilicata, Calabria, Sicily and Sardinia. Only recently, such organizations were merged into two main interregional areas (see Fig.3.4).

- *National coordination level.* It consists of the Centro Nazionale Trapianti which coordinates the explants and the transplants throughout the national territory, manages the waiting lists and guarantees the quality of services. Also it deals with the organ allocation and is supported by the Consulta Tecnica Permanente.

The National Transplant Network, which was designed about 20 years ago and was progressively renewed in some respects, has undergone a major transformation from the 4-tier system described above to a 3-level system of coordination (national, regional, local/hospital) by means of the institution, at the Istituto Superiore di Sanità, of the National Center for Operational Transplantation (CNTO). The CNTO manages all the emergencies, the reports from Italian donors, all the national transplant programs and the exchanges with European countries. Thanks to the system's national vision and to the synergies with the Regional Centers, the CNTO has made the compensation system for the organ restraint program more efficient and slender, allowing for significantly reducing the number of transports carried out by the surgical teams and generating a total cost-effective annual savings of at least one million euros over the average of previous years.

For the success of a transplant it is fundamental an excellent management of the coordination centers. To understand the complex system managed by these centers, we analyze in detail the donation-transplant process, which begins with the identification of a potential donor and ends with the transplant itself.

In some cases, such as in kidney and liver transplants, living donors can offer their organs for transplants (see, for instance, [6] and [61]), but in our model we assume that it is necessary to determine the brain death of the patient before working with the removal of his/her organs.
In Italy the mean time which elapses from the reporting of the donor patient to the transplant surgery is approximately 10 hours. During this time there are about 100 people from different sectors and structures interacting with the donation-transplant process. In this period, it is necessary to certify the death according to the criteria established by the law, to verify the suitability of the donor and his organs, to consult the waiting lists, to identify the possible recipients, to check the compatibility between the donor and the recipient, to allocate the available organs to selected patients while guaranteeing emergencies, to identify the most suitable transport for biological samples and medical teams, to call recipients in transplant centers, to remove and to transplant the organs, and ultimately to take care of the patient during the postoperative phase. The total donation-transplant

process involves different structures (hospitals, coordination centers, immunology laboratories, transplant centers, transport companies, etc.) and skills (medical, surgical, logistics, etc.) who must be able to interact each other for the success of the transplant.

We now describe the main stages of the donation-transplant process.
The first important step is the *identification of the potential donor*. To this end, it is necessary to monitor the subjects in the health facilities. Hence, the partnership with the Health Department and with the diagnostic imaging services is fundamental. In addition, very important are also the campaigns promoting organ donation in order to inform citizens about the willingness to donate, to raise awareness on the riability of the Italian system for transplants in terms of transparency of the criteria for inclusion on the waiting list, of the safety of the explant and transplant procedures and of death verification rules.
Official data confirm an increase of transplants and donations in Italy. In 2015 a total of 3317 organ transplants have been operated (67 more than 2014 and 228 more than 2013). The entire transplant activity is growing and the percentage of oppositions to donation in 2015 is decreasing, reaching 30.6 % instead of 31 %.

An extremely delicate phase concerns *the diagnosis, the assessment and the certification of death*. In subjects suffering from encephalic lesions who satisfy the conditions laid down by the law (no. 578 - December 29, 1993 and no. 582 - August 22, 1994), the doctor of the hospital which is in charge of the subject must immediately communicate to the Health Department the existence of a case of death with irreversible cessation of all brain functions. Then, a medical team is activated, consisting of a resuscitator, a forensic doctor and a neurologist, who will proceed with the death certification by neurological criteria.

The *indication of the potential donor to the Regional Transplant Center* is done by the resuscitator or by the local coordinator of the structure that is in charge of the subject. All the available information, such as cause of death, age, medical history, blood type, and so on, is then transmitted. A *first fitness assessment* of the potential donor with a series of medical tests follows. To avoid alterations and to ensure a good functionality of the organs, the subject's brain-dead must be preserved by the resuscitator.

The most delicate phase is surely the *interview with the potential donor's family*. The dialogue must be clear, transparent and consistent, and the news of the death must always precede the donation proposal.

Simultaneously, the *consultation of waiting lists and the allocation of organs* take place. The consultation is carried out by the Regional and/or Interregional Centre which manages the waiting lists. Each transplant center transmits to its Regional Centre any new patient inclusion as well as any clinical update of the patients already included in the waiting list. Each regional or interregional center may use its own allocation algorithm, whose criteria can be different in the different regions, but satisfy common principles. Generally the same centers take charge of the organ removal operations. In addition, after checking the presence of priorities on a national basis, the Regional Centre of the donor allocates available organs in its own region.

The *organ removal* is a critical step for the success of the transplant. The medical teams may come from different centers, and are generally expected in the operating room by the local coordinator who guides them throughout the entire process of explant. In addition, such teams are required to make a *second fitness evaluation*. Each team checks for all the necessary tools and provides independently the material for the storage and the organ transport.

The last phase for the success of a transplant is a careful surgical preparation of the organ, which allows a final fitness evaluation and the correction of any anatomical abnormalities.

The main part of the whole process is the *transplant*, a complex surgical procedure that consists in removing a diseased organ and replacing it with a healthy organ. It is a moment of great responsibility for the entire transplant system. The surgery duration varies depending on the organ between 2 up to 15 hours. Not only the medical team is involved in the transplant, but the entire transplant center. Not less important is the phase of *follow-up*, i.e. the planning and the execution of controls on the patient after the transplant operation and for all the time necessary for the complete stabilization of the clinical conditions.

An important aspect throughout all the donation-transplant process is the *logistics*. Indeed it is used to plan and to coordinate all activities which are necessary to reach the goal as quickly as possible. These activities are concerned with the medical team, the organs, and the biological materials transport. Logistics is surely one of the most important aspects because an improper management can lead to delays and problems during the whole process. As a consequence, vehicles and aircrafts must be available next to every transplant center.

The theoretical contribution of the paper consists in formulating the network model associated to the translpant process and to find the optimality conditions for the minimization of the total related costs which are charged by the National Health Service. Moreover, we are able to express these equilibrium conditions in terms of a variational inequality formulation.

Notably, the Lagrange multipliers associated with the constraints allow for an elegant economic interpretation.

The paper is organized as follows. In Section 3.2 we present the organ transplant network consisting of transplant centers and donor hospitals. We introduce the cost functions associated with transportations, with organ removals, with waste disposals, and with post-transplants. We determine the optimality conditions for the national health service and derive the variational inequality formulation. In Section 3.3 we study the Lagrange theory related to the model in order to better understand the behavior of the transplant process, providing an interpretation of the Lagrange multipliers. In Section 3.4 we recall the Euler method which has been applied in Section 3.5 to solve numerical examples. Section 3.6 is devoted to the conclusions.

3.2 The mathematical model

The organ transplant network we are examining consists of m transplant centers, with a typical one denoted by i, and n donor hospitals, with a typical one denoted by j. In addition, there are v different transportation services given, such as ambulance, airplane, helicopter, etc., with a typical one denoted by k. In the network we model the different transportation services as parallel links connecting a given transplant center node to a given donor hospital node and viceversa.

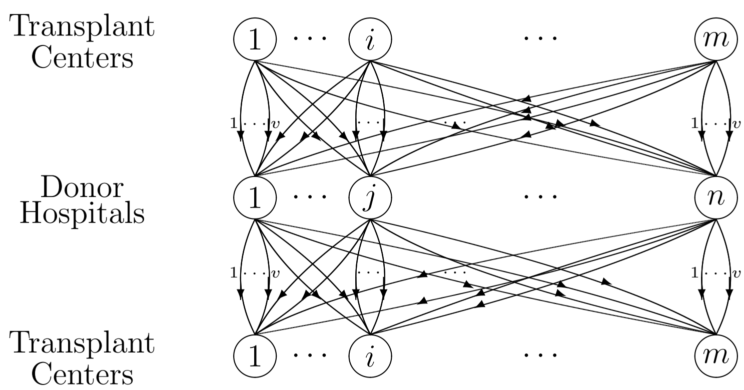

Figure 3.3: Organ transport network

In Figure 3, the underlying network structure of the optimization problem is depicted. The nodes at the highest and at the lowest levels represent the transplant centers, the intermediate level nodes stand for the donor hospitals. The uppermost links, in turn, correspond to the medical teams transportation whereas the lowermost links correspond both to the organs and to the medical teams transportation. Such a network topology arises from the real situation. Indeed, once an organ is available at a donor hospital, the teams of transplant centers take charge of its removal and transplant.

Let g_{ijk} be the quantity of medical teams moving from the transplant center i to the donor hospital j using the $k-$th transportation service and we group such flows into the vector $G^1 \in \mathbb{R}^{mnv}_+$. Let c^{TE}_{ijk} be the transportation costs associated to the medical teams from the transplant center i to the donor hospital j using the $k-$th transportation service and we assume c^{TE}_{ijk} as a function of g_{ijk}:

$$c^{TE}_{ijk} = c^{TE}_{ijk}(g_{ijk}), \quad \forall i = 1, \dots, m, \forall j = 1, \dots, n, \forall k = 1, \dots, v.$$

Let g_j be the quantity of organs available at the donor hospital j and we group such quantities into the vector $G^2 \in \mathbb{R}^n_+$. Let c^S_j be the health costs due to the organ removal at the hospital j and we assume such costs as a function of g_j:

$$c^S_j = c^S_j(g_j), \quad \forall j = 1, \dots, n.$$

Let $\tilde{g}_{jik}$ be the quantity of organs sent from the donor hospital j to the transplant center i using the $k-$th transportation service and we group such flows into the vector $G^3 \in \mathbb{R}^{nmv}_+$. Let c^{TO}_{jik} be the transportation costs associated to the organs from the donor hospital j to the transplant center i using the $k-$th transportation service and we assume c^{TO}_{jik} as a function of $\tilde{g}_{jik}$:

$$c^{TO}_{jik} = c^{TO}_{jik}(\tilde{g}_{jik}), \quad \forall j = 1, \dots, n, \forall i = 1, \dots, m, \forall k = 1, \dots, v.$$

Let $\tilde{g}_i$ be the quantity of organs transplanted at the center i and we group such quantities into the vector $G^4 \in \mathbb{R}^m_+$. Let $\tilde{c}^S_i$ be the health costs due to the transplant at the center i and we assume such costs as a function of $\tilde{g}_i$:

$$\tilde{c}^S_i = \tilde{c}^S_i(\tilde{g}_i), \quad \forall i = 1, \dots, m.$$

Let c^{POST}_i the costs incurred in the center i during the post-transplant and we assume they are a function of $\tilde{g}_i$:

$$c^{POST}_i = c^{POST}_i(\tilde{g}_i), \quad \forall i = 1, \dots, m.$$

Let c^W_j be the unit special waste disposal cost at the donor hospital j (for instance, explanted organs which are unfit for transplant). Let $\beta_j \in [0, 1]$ be the portion of explanted organs discarded in the donor hospital j. Further, let $\tilde{c}_i^W$ be the unit special waste disposal cost at the transplant center i (for instance, diseased organs to be replaced).

We recall that every organ has a cold ischemia time. Specifically, "cold ischemia time during organ transplantation begins when the organ is cooled with a cold perfusion solution after organ procurement surgery, and ends after the tissue reaches physiological temperature during implantation procedures" (http://www.reference.md/files/D050/mD050377.html). Hence, let $\gamma_i \in [0, 1]$ be the portion of organs reaching the transplant center i, but which cannot be transplanted because of delays in the transportation (exceeding the cold ischemia time) or because they result to be not compatible with the recipient patients. As a consequence, in every transplant center i the quantities of organs which must be wasted is given by $\tilde{g}_i + \gamma_i \sum_{j=1}^{n} \sum_{k=1}^{v} \tilde{g}_{jik}$, as well as the relationship among $\tilde{g}_i$, $\tilde{g}_{jik}$ and γ_i is given by:

$$\tilde{g}_i = (1 - \gamma_i) \sum_{j=1}^{n} \sum_{k=1}^{v} \tilde{g}_{jik},$$

namely the number of transplanted organs is the same as the number of transported organs minus the wasted ones.

Finally, we denote by $\tilde{c}_i^W$ and c_j^W the unit special waste disposal costs at the transplant center i, $i = 1, \ldots, m$ and at the donor hospital j, $j = 1, \ldots, n$, respectively.

In this model our aim is to minimize the total costs incurred by the National Health Service for transplants. So the optimality conditions are as follows:

$$\min \left\{ \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{v} 2c_{ijk}^{TE}(g_{ijk}) + \sum_{j=1}^{n} c_j^S(g_j) + \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{v} c_{jik}^{TO}(\tilde{g}_{jik}) \right.$$
$$+ \sum_{i=1}^{m} \tilde{c}_i^S(\tilde{g}_i) + \sum_{i=1}^{m} c_i^{POST}(\tilde{g}_i) + \sum_{j=1}^{n} \beta_j c_j^W g_j \tag{3.1}$$
$$\left. + \sum_{i=1}^{m} \tilde{c}_i^W \left(\tilde{g}_i + \gamma_i \sum_{j=1}^{n}\sum_{k=1}^{v} \tilde{g}_{jik} \right) \right\}$$

subject to the constraints:

$$g_{ijk} \geq 0 \quad \forall i = 1, \ldots, m, \forall j = 1, \ldots, n, \forall k = 1, \ldots, v; \tag{3.2}$$

$$g_j \geq 0 \quad \forall j = 1, \ldots, n; \tag{3.3}$$

$$\tilde{g}_{jik} \geq 0 \quad \forall j = 1, \ldots, n, \forall i = 1, \ldots, m, \forall k = 1, \ldots, v; \tag{3.4}$$

$$\tilde{g}_i \geq 0 \quad \forall i = 1, \ldots, m; \tag{3.5}$$

$$\sum_{i=1}^{m}\sum_{k=1}^{v} g_{ijk} \leq g_j \quad \forall j = 1, \ldots, n; \tag{3.6}$$

$$\tilde{g}_i \leq \sum_{j=1}^{n} (1 - \beta_j) g_j \quad \forall i = 1, \ldots, m; \tag{3.7}$$

$$\tilde{g}_i \leq \sum_{j=1}^{n}\sum_{k=1}^{v} \tilde{g}_{jik} \quad \forall i = 1, \ldots, m; \tag{3.8}$$

$$\tilde{g}_i = (1 - \gamma_i) \sum_{j=1}^{n}\sum_{k=1}^{v} \tilde{g}_{jik} \quad \forall i = 1, \ldots, m; \tag{3.9}$$

$$\sum_{i=1}^{m}\sum_{k=1}^{v} \tilde{g}_{jik} = (1 - \beta_j)\, g_j \quad \forall j = 1, \ldots, n. \tag{3.10}$$

It is worth remarking that the objective function in (3.1) does not attain its minimum value in $(G^{1*}, G^{2*}, G^{3*}, G^{4*}) = (0, 0, 0, 0)$, since the marginal cost functions (which are greatly influenced by the fixed costs) decrease when a large number of transplants is performed.

Constraints (3.2), (3.3), (3.4) and (3.5) ensure that the flows of medical teams, of available organs, of transported organs and of transplanted organs, respectively, are nonnegative.
Constraint (3.6) ensures that the number of medical teams moving from every transplant center and using every transmission service does not exceed the number of available organs at the donor hospital.
Constraint (3.7) states that the number of transplanted organs does not exceed the sum of the portions of available organs in all donor hospitals which are not discarded.
Constraint (3.8) states that the number of transplanted organs does not exceed the number of transported ones, for every transplant center.
Constraint (3.9) ensures that the number of transplanted organs is the same as the number of transported organs minus the wasted ones.
Finally, constraint (3.10) ensures that at any donor hospital j the explanted organs which are not discarded are exactly the same as the ones transported from j to every transplant center using every transmission service.
Problem (3.1) can be characterized by the following variational inequality:

Find $(G^{1*}, G^{2*}, G^{3*}, G^{4*}) \in \mathbb{K}$ such that:

$$\begin{aligned}
&\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{v} 2\frac{\partial c_{ijk}^{TE}(g_{ijk}^*)}{\partial g_{ijk}} \times [g_{ijk} - g_{ijk}^*] \\
&+\sum_{j=1}^{n}\left[\frac{\partial c_j^S(g_j^*)}{\partial g_j} + \beta_j c_j^W\right] \times [g_j - g_j^*] \\
&+\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{v}\left[\frac{\partial c_{jik}^{TO}(\tilde{g}_{jik}^*)}{\partial \tilde{g}_{jik}} + \gamma_i \tilde{c}_i^W\right] \times [\tilde{g}_{jik} - \tilde{g}_{jik}^*] \\
&+\sum_{i=1}^{m}\left[\frac{\partial \tilde{c}_i^S(\tilde{g}_i^*)}{\partial \tilde{g}_i} + \frac{\partial c_i^{POST}(\tilde{g}_i^*)}{\partial \tilde{g}_i} + \tilde{c}_i^W\right] \times [\tilde{g}_i - \tilde{g}_i^*] \geq 0, \\
&\qquad \forall (G^1, G^2, G^3, G^4) \in \mathbb{K},
\end{aligned} \tag{3.11}$$

where:

$$\begin{aligned}\mathbb{K} \;=\; & \Bigg\{(G^1, G^2, G^3, G^4) \in \mathbb{R}_+^{mnv+n+nmv+m} : \\ & \sum_{i=1}^{m}\sum_{k=1}^{v} g_{ijk} \le g_j \quad \forall j = 1, \dots, n; \\ & \tilde{g}_i \le \sum_{j=1}^{n}(1-\beta_j)g_j \quad \forall i = 1, \dots, m; \\ & \tilde{g}_i \le \sum_{j=1}^{n}\sum_{k=1}^{v} \tilde{g}_{jik} \quad \forall i = 1, \dots, m; \\ & \tilde{g}_i = (1-\gamma_i)\sum_{j=1}^{n}\sum_{k=1}^{v} \tilde{g}_{jik} \quad \forall i = 1, \dots, m; \\ & \sum_{i=1}^{m}\sum_{k=1}^{v} \tilde{g}_{jik} = (1-\beta_j)g_j \quad \forall j = 1, \dots, n\Bigg\}.\end{aligned}$$

Now we make the following assumptions:

Hp.1 *Let all the involved functions (such as the transportation costs, the health costs, the post-transplant costs, the special waste disposal costs) be continuously differentiable and convex.*

Variational inequality (3.11) can be put in a standard form (see [42]) as follows:

Find $X^* \in \mathcal{K}$ such that:

$$\langle F(X^*), X - X^* \rangle \ge 0 \quad \forall X \in \mathcal{K}, \tag{3.12}$$

where:

- $\langle \cdot, \cdot \rangle$ denotes the inner product in the $mnv+n+nmv+m$- dimensional Euclidean space;
- $\mathcal{K} \equiv \mathbb{K}$;
- $X \equiv (G^1, G^2, G^3, G^4)$;

- $F(X) \equiv (F_1(X), F_2(X), F_3(X), F_4(X))$, where

$$\begin{aligned}
F_1(X) &= \left[2\frac{\partial c^{TE}_{ijk}(g^*_{ijk})}{\partial g_{ijk}}; i=1,\dots,m,\ j=1,\dots,n,\ k=1,\dots,v\right],\\
F_2(X) &= \left[\frac{\partial c^S_j(g^*_j)}{\partial g_j} + \beta_j c^W_j; j=1,\dots,n\right]\\
F_3(X) &= \left[\frac{\partial c^{TO}_{jik}(\tilde{g}^*_{jik})}{\partial \tilde{g}_{jik}} + \gamma_i \tilde{c}^W_i; j=1,\dots,n,\ i=1,\dots,m,\ k=1,\dots,v\right],\\
F_4(X) &= \left[\frac{\partial \tilde{c}^S_i(\tilde{g}^*_i)}{\partial \tilde{g}_i} + \frac{\partial c^{POST}_i(\tilde{g}^*_i)}{\partial \tilde{g}_i} + \tilde{c}^W_i; i=1,\dots,m\right].
\end{aligned}$$

Since the feasible set $\mathbb{K}$ is closed and convex, because of constraints (3.2)-(3.10), we can obtain the existence of a solution to (3.12) based on the assumption of the continuity of F and requiring a coercivity condition.

Therefore, following [31], we have the following theorem:

Theorem 5 (Existence)**.** *Let us assume that assumptions* (**Hp.1**) *are satisfied and that the following coercivity assumption is fulfilled:*

$$\lim_{\substack{\|X\|\to\infty \\ X\in\mathbb{K}}} \frac{\langle F(X), X\rangle}{\|X\|} = \infty.$$

Then, there exists at least one solution to variational inequality (3.12).

In addition, we now provide a uniqueness result.

Theorem 6 (Uniqueness)**.** *Under the assumptions of Theorem 5, if the function $F(X)$ in (3.12) is strictly monotone on $\mathcal{K}$, that is:*

$$\langle \left(F(X^1) - F(X^2)\right)^T, X^1 - X^2\rangle > 0, \quad \forall X^1, X^2 \in \mathcal{K},\ X^1 \neq X^2,$$

then variational inequality (3.12) and, hence, variational inequality (3.11), admits a unique solution.

3.3 Lagrange Theory

In this section we explore the Lagrange theory associated with variational inequality (3.11), so that we better understand the behavior of the transplant process (see also [3], [13], [14], [15], [16], [28], [18], [49], [63], for an application of the Lagrange theory to financial networks, to spatial economic models, to random traffic networks, to elastic-plastic torsion problems, to cybersecurity investment supply chain game theory models, and to end-of-life products networks, respectively). To this aim, we set:

$$
\begin{aligned}
a_j &= \sum_{i=1}^{m}\sum_{k=1}^{v} g_{ijk} - g_j \le 0, && \forall j = 1,\dots,n;\\
b_i &= \tilde{g}_i - \sum_{j=1}^{n}(1-\beta_j) g_j \le 0, && \forall i = 1,\dots,m;\\
p_i &= \tilde{g}_i - \sum_{j=1}^{n}\sum_{k=1}^{v} \tilde{g}_{jik} \le 0 && \forall i = 1,\dots,m;\\
d_j &= (1-\beta_j) g_j - \sum_{i=1}^{m}\sum_{k=1}^{v} \tilde{g}_{jik} = 0, && \forall j = 1,\dots,n;\\
q_i &= \tilde{g}_i - (1-\gamma_i)\sum_{j=1}^{n}\sum_{k=1}^{v} \tilde{g}_{jik}, && \forall i = 1,\dots,m;\\
e_{ijk} &= -g_{ijk} \le 0, && \forall i = 1,\dots,m,\ \forall j = 1,\dots,n,\ \forall k = 1,\dots,v;\\
f_j &= -g_j \le 0, && \forall j = 1,\dots,n;\\
h_{jik} &= -\tilde{g}_{jik} \le 0 && \forall j = 1,\dots,n,\ \forall i = 1,\dots,m,\ \forall k = 1,\dots,v;\\
l_i &= -\tilde{g}_i \le 0, && \forall i = 1,\dots,m;
\end{aligned}
$$

and

$$\Gamma(X) = (a_j, b_i, p_i, d_j, q_i, e_{ijk}, f_j, h_{jik}, l_i)_{i=1,\dots,m,\, j=1,\dots,n,\, k=1,\dots,v}\,.$$

As a consequence, we remark that $\mathbb{K}$ can be rewritten as

$$\mathbb{K} = \left\{ X \in \mathbb{R}_+^{2mnv+n+m} : \Gamma(X) \le 0 \right\}.$$

We can now consider the following Lagrange function:

$$\begin{aligned}\mathcal{L}(X,\omega,\varphi,\vartheta,\psi,\varepsilon,\lambda,\nu,\mu,\sigma) &= \langle F(X^*), X-X^*\rangle + \sum_{j=1}^{n}\omega_j a_j + \sum_{i=1}^{m}\varphi_i b_i \\ &+ \sum_{i=1}^{m}\vartheta_i p_i + \sum_{j=1}^{n}\psi_j d_j + \sum_{i=1}^{m}\varepsilon_i q_i + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{v}\lambda_{ijk}e_{ijk} \\ &+ \sum_{j=1}^{n}\nu_j f_j + \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{v}\mu_{jik}h_{jik} + \sum_{i=1}^{m}\sigma_i l_i \qquad (3.13)\end{aligned}$$

$$\forall X \in \mathbb{R}_+^{2mnv+n+m},\ \forall\omega \in \mathbb{R}_+^n,\ \forall\varphi \in \mathbb{R}_+^m,\ \forall\vartheta \in \mathbb{R}_+^m,\ \forall\psi \in \mathbb{R}^n,\ \forall\varepsilon \in \mathbb{R}^m,$$

$$\forall\lambda \in \mathbb{R}_+^{mnv},\ \forall\nu \in \mathbb{R}_+^n,\ \forall\mu \in \mathbb{R}_+^{nmv},\ \forall\sigma \in \mathbb{R}_+^m.$$

Variational inequality (3.12) can be written as:

$$\min_{X\in\mathbb{K}}\langle F(X^*), X-X^*\rangle = 0. \qquad (3.14)$$

Indeed, we have $\langle F(X^*), X-X^*\rangle \geq 0$ in $\mathbb{K}$ and $\langle F(X^*), X^*-X^*\rangle = 0$.
Its dual problem is:

$$\max_{(\Pi,\psi,\varepsilon)\in\mathbb{R}_+^{2n+3m+2mnv}\times\mathbb{R}^n\times\mathbb{R}^m}\ \inf_{X\in\mathbb{R}_+^{2mnv+n+m}} \mathcal{L}(X,\Pi,\psi,\varepsilon) \qquad (3.15)$$

where $(\Pi,\psi,\varepsilon) = (\omega,\varphi,\vartheta,\lambda,\nu,\mu,\sigma,\psi,\varepsilon)$. We recall that the problem of the strong duality between (3.14) and (3.15) is to find

$$\min_{X\in\mathbb{K}}\langle F(X^*), X-X^*\rangle = \max_{(\Pi,\psi,\varepsilon)\in\mathbb{R}_+^{2n+3m+2mnv}\times\mathbb{R}^n\times\mathbb{R}^m}\ \inf_{X\in\mathbb{R}_+^{2mnv+n+m}} \mathcal{L}(X,\Pi,\psi,\varepsilon). \qquad (3.16)$$

The following result holds true.

Theorem 7. *Problem (3.14) satisfies the Karush-Khun-Tucker conditions.*

Proof Let us recall that the KKT conditions for the existence of the Lagrange multipliers can be written as follows (see [29]). Let X^* be the solution to variational inequality (3.14) and let us set:

$$
\begin{aligned}
I_{a_j}(X^*) &= \{j \in \{1,\ldots,n\} : a_j = 0\}; \\
I_{b_i}(X^*) &= \{i \in \{1,\ldots,m\} : b_i = 0\}; \\
I_{p_i}(X^*) &= \{i \in \{1,\ldots,m\} : p_i = 0\}; \\
I_{e_{ijk}}(X^*) &= \{(i,j,k) \in \{1,\ldots,m\} \times \{1,\ldots,n\} \times \{1,\ldots,v\} : e_{ijk} = 0\}; \\
I_{f_j}(X^*) &= \{j \in \{1,\ldots,n\} : f_j = 0\}; \\
I_{h_{jik}}(X^*) &= \{(j,i,k) \in \{1,\ldots,n\} \times \{1,\ldots,m\} \times \{1,\ldots,v\} : h_{ijk} = 0\}; \\
I_{l_i}(X^*) &= \{i \in \{1,\ldots,m\} : l_i = 0\}.
\end{aligned}
$$

Then the existence of the Lagrange multipliers is ensured if there exists a vector $X \in \mathbb{R}_+^{2mnv+n+m}$ such that the KKT conditions are verified, namely:

$$
\begin{aligned}
\sum_{i=1}^{m}\sum_{k=1}^{v} g_{jik} - g_j < 0, &\qquad \forall j \in I_{a_j}(X^*); \\
\tilde{g}_i - \sum_{j=1}^{n}(1-\beta_j)g_j < 0, &\qquad \forall i \in I_{b_i}(X^*); \\
\tilde{g}_i - \sum_{j=1}^{n}\sum_{k=1}^{v} \tilde{g}_{jik} < 0, &\qquad \forall i \in I_{p_i}(X^*); \\
\tilde{g}_i - (1-\gamma_i)\sum_{j=1}^{n}\sum_{k=1}^{v} \tilde{g}_{jik} = 0, &\qquad \forall i = 1,\ldots,m; \qquad (3.17)
\end{aligned}
$$

$$
\begin{aligned}
(1-\beta_j)g_j - \sum_{i=1}^{m}\sum_{k=1}^{v} \tilde{g}_{jik} = 0, &\qquad \forall j = 1,\ldots,n; \\
-g_{ijk} < 0, &\qquad \forall (i,j,k) \in I_{e_{ijk}}(X^*); \\
-g_j < 0, &\qquad \forall j \in I_{f_j}(X^*); \\
-\tilde{g}_{jik} < 0 &\qquad \forall (j,i,k) \in I_{h_{jik}}(X^*); \\
-\tilde{g}_i < 0, &\qquad \forall i \in I_{l_i}(X^*).
\end{aligned}
$$

It is easy to verify that the systems (3.17) admits a solution and that the vectors ∇d_j, $j = 1,\ldots,n$, and ∇q_i, $i = 1,\ldots,m$ are linearly independent. □

As a consequence, we get the following result.

Theorem 8. *Let X^* be the solution to variational inequality (3.12), then the Lagrange multipliers $\omega^* \in \mathbb{R}^n_+$, $\varphi^* \in \mathbb{R}^m_+$, $\vartheta^* \in \mathbb{R}^m_+$, $\psi^* \in \mathbb{R}^n$, $\varepsilon^* \in \mathbb{R}^m$, $\lambda^* \in \mathbb{R}^{mnv}_+$, $\nu^* \in \mathbb{R}^n_+$, $\mu^* \in \mathbb{R}^{nmv}_+$, $\sigma^* \in \mathbb{R}^m_+$ associated with the constraints $a_j \leq 0$, $b_i \leq 0$, $p_i \leq 0$, $d_j = 0$, $q_i = 0$, $e_{ijk} \leq 0$, $f_i \leq 0$, $h_{jik} \leq 0$, $l_i \leq 0$ do exist.*

Also the following result holds true.

Theorem 9. *Let us assume that assumptions* (**Hp.1**) *are satisfied. Then a vector $X^* \in \mathbb{K}$ is a solution to variational inequality (3.12) if and only if the vector $(X^*, \omega^*, \varphi^*, \vartheta^*, \psi^*, \varepsilon^*, \lambda^*, \nu^*, \mu^*, \sigma^*)$ is a saddle point of the Lagrange function (3.13), namely:*

$$\begin{aligned} \mathcal{L}(X^*, \omega, \varphi, \vartheta, \psi, \varepsilon, \lambda, \nu, \mu, \sigma) &\leq \mathcal{L}(X^*, \omega^*, \varphi^*, \vartheta^*, \psi^*, \varepsilon^*, \lambda^*, \nu^*, \mu^*, \sigma^*) \\ &\leq \mathcal{L}(X, \omega^*, \varphi^*, \vartheta^*, \psi^*, \varepsilon^*, \lambda^*, \nu^*, \mu^*, \sigma^*) \end{aligned} \quad (3.18)$$

$$\forall X \in \mathbb{K},\ \forall \omega \in \mathbb{R}^n_+,\ \forall \varphi \in \mathbb{R}^m_+,\ \forall \vartheta \in \mathbb{R}^m_+,\ \forall \psi \in \mathbb{R}^n,\ \forall \varepsilon \in \mathbb{R}^m,$$

$$\forall \lambda \in \mathbb{R}^{mnv}_+,\ \forall \nu \in \mathbb{R}^n_+,\ \forall \mu \in \mathbb{R}^{nmv}_+,\ \forall \sigma \in \mathbb{R}^m_+,$$

and

$$\begin{array}{llll} \omega^*_j a^*_j = 0, & \nu^*_j f^*_j = 0, & & \forall j; \\ \vartheta^*_i p^*_i = 0, & \sigma^*_i l^*_i = 0, & \varphi^*_i b^*_i = 0, & \forall i; \\ \lambda^*_{ijk} e^*_{ijk} = 0, & \mu^*_{jik} h^*_{jik} = 0, & & \forall i,\ \forall j,\ \forall v. \end{array} \quad (3.19)$$

Proof See Theorem 5 in [17], since the KKT conditions imply that Assumption S is verified. □

In virtue of Theorem 9, we can calculate the Lagrange multipliers $\omega^* \in \mathbb{R}^n_+$, $\varphi^* \in \mathbb{R}^n_+$, $\psi^* \in \mathbb{R}^n_+$, $\varepsilon^* \in \mathbb{R}^m_+$, $\lambda^* \in \mathbb{R}^{mnv}_+$, $\nu^* \in \mathbb{R}^n_+$, $\mu^* \in \mathbb{R}^{nmv}_+$, $\sigma^* \in \mathbb{R}^m_+$ associated with the constraints and the solution X^* to variational inequality (3.12).

From the right-hand side of (3.18) it follows that $X^* \in \mathbb{R}^{2mnv+n+m}$ is a minimal point of $\mathcal{L}(X, \omega^*, \varphi^*, \psi^*, \varepsilon^*, \lambda^*, \nu^*, \mu^*, \sigma^*)$ in the whole space $\mathbb{R}^{2mnv+n+m}$ and hence, for all $i = 1, \dots, m$, $j = 1, \dots, n$, $k = 1, \dots, v$, we get:

$$\frac{\partial \mathcal{L}(X^*, \omega^*, \varphi^*, \vartheta^*, \psi^*, \varepsilon^*, \lambda^*, \nu^*, \mu^*, \sigma^*)}{\partial g_{ijk}} = 2\frac{\partial c_{ijk}^{TE}(g_{ijk}^*)}{\partial g_{ijk}} + \omega_j^* - \lambda_{ijk}^* = 0, \quad (3.20)$$

$$\frac{\partial \mathcal{L}(X^*, \omega^*, \varphi^*, \vartheta^*, \psi^*, \varepsilon^*, \lambda^*, \nu^*, \mu^*, \sigma^*)}{\partial g_j} = \frac{\partial c_j^S(g_j^*)}{\partial g_j} + \beta_j c_j^W - (1-\beta_j)\varphi_i^* + (1-\beta_j)\psi_i^* - \nu_j^* = 0, \quad (3.21)$$

$$\frac{\partial \mathcal{L}(X^*, \omega^*, \varphi^*, \vartheta^*, \psi^*, \varepsilon^*, \lambda^*, \nu^*, \mu^*, \sigma^*)}{\partial \tilde{g}_{jik}} = \frac{\partial c_{jik}^{TO}(\tilde{g}_{jik}^*)}{\partial \tilde{g}_{jik}} + \gamma_i \tilde{c}_i^W - \vartheta_i^* - \psi_i^* - (1-\gamma_i)\,\varepsilon_i^* - \mu_{jik}^* = 0, \quad (3.22)$$

$$\frac{\partial \mathcal{L}(X^*, \omega^*, \varphi^*, \vartheta^*, \psi^*, \varepsilon^*, \lambda^*, \nu^*, \mu^*, \sigma^*)}{\partial \tilde{g}_i} = \frac{\partial \tilde{c}_i^S(\tilde{g}_i^*)}{\partial \tilde{g}_i} + \frac{\partial c_i^{POST}(\tilde{g}_i^*)}{\partial \tilde{g}_i} + \tilde{c}_i^W + \vartheta_i^* + \varepsilon_i^* + \varphi_i^* + \sigma_i^* = 0, \quad (3.23)$$

together with conditions (3.19).
It is easy to see that conditions (3.19)-(3.23) are equivalent to variational inequality (3.11).

The importance of the Lagrange function consists in the fact that constraints are included in such a function and it allows us, when the strong duality holds, to express the solution to the variational inequality by means of the system of equations derived from the KKT conditions. The existence of the solution to the variational inequality is guaranteed by Theorem 5.

Remark 1.

Now we can interpret the meaning of some Lagrange variables. Let us consider, first, the case when $\omega_j^* > 0$. Then, from (3.19), we get:

$$a_j^* = 0 \Longleftrightarrow \sum_{i=1}^{m}\sum_{k=1}^{v} g_{ijk}^* = g_j^*,$$

which means that the number of medical teams reaching the donor hospital j equals the number of available organs in j. Also, let us assume that all the medical teams are active, namely $g^*_{ijk} > 0$ which implies, from (3.19), that $\lambda^*_{ijk} = 0$. Hence (3.20) becomes:

$$\frac{\partial c^{TE}_{ijk}(g^*_{ijk})}{\partial g_{ijk}} = -\frac{\omega^*_j}{2} < 0.$$

On the contrary, if $\sum_{i=1}^{m}\sum_{k=1}^{v} g^*_{ijk} < g^*_j$ and $g^*_{ijk} > 0$, then, from (3.19), we get $\omega^*_j = 0$, and $\lambda^*_{ijk} = 0$. Hence, (3.20) becomes (although we use the same symbol g^*_{ijk} for the solutions, g^*_{ijk} assumes different values for the different cases under consideration):

$$\frac{\partial c^{TE}_{ijk}(g^*_{ijk})}{\partial g_{ijk}} = 0.$$

So, in the last case, in g^*_{ijk} the cost function c^{TE}_{ijk} attains the minimum value, which means that, when the number of medical teams reaching the donor hospital j is less than the number of available organs in j, the transport costs for the medical teams reach the minimum values.
Analogously, from (3.19), we get:

$$\varphi^*_i\left(\tilde{g}^*_i - \sum_{j=1}^{n}(1-\beta_j)g^*_j\right) = 0 \text{ and } \nu^*_j(-g^*_j) = 0 \qquad (3.24)$$

If we assume that there available organs at the donor hospital j, namely $g^*_j > 0$, then $\nu^*_j = 0$. Also, if we assume $\tilde{g}^*_i < \sum_{j=1}^{n}(1-\beta_j)g^*_j$, then $\varphi^*_i = 0$. Hence (3.21) becomes:

$$\frac{\partial c^S_j(g^*_j)}{\partial g_j} + \beta_j c^W_j = -(1-\beta_j)\psi^*_i, \quad \forall j = 1,\dots,n.$$

Summing up with respect to j, we obtain:

$$\sum_{j=1}^{n}\left(\frac{\partial c^S_j(g^*_j)}{\partial g_j} + \beta_j c^W_j\right) = -\sum_{j=1}^{n}(1-\beta_j)\psi^*_i.$$

As a consequence, $-\sum_{j=1}^{n}(1-\beta_j)\psi^*_i$ represents the sum of the marginal costs associated to the explantation and to the special waste disposal. On the contrary, if $\varphi_i^* > 0$, then $\tilde{g}_i^* = \sum_{j=1}^{n}(1-\beta_j)g_j^*$ and (3.21) becomes:

$$\frac{\partial c_j^S(g_j^*)}{\partial g_j} + \beta_j c_j^W - (1-\beta_j)\varphi_i^* = -(1-\beta_j)\psi_i^*, \quad \forall j = 1,\ldots,n,$$

which yields:

$$\sum_{j=1}^{n}\left(\frac{\partial c_j^S(g_j^*)}{\partial g_j} + \beta_j c_j^W\right) = \sum_{j=1}^{n}(1-\beta_j)\left(\varphi_i^* - \psi_i^*\right).$$

Hence, the total marginal cost increases, since a greater number of transplants is performed.
Analogous considerations hold for (3.22) and (3.23).

3.4 The algorithm

We now recall the Euler method (see the general scheme in [21]). For every iteration k, we calculate:

$$X^{k+1} = P_{\mathcal{K}}\left(X^k - a_k F\left(X^k\right)\right), \tag{3.25}$$

where $P_{\mathcal{K}}$ is the projection on the feasible set $\mathcal{K}$ defined as

$$P_{\mathcal{K}}(\xi) = \underset{z\in\mathcal{K}}{\operatorname{argmin}}\,||\xi - z||$$

and F is the function that enters the variational inequality problem (3.12)). In order to get the convergence of the iterative scheme, we need the sequence $\{a_k\}$ to be such that:

$$\sum_{k=0}^{\infty} a_k = \infty,\ a_k > 0,\ a_k \to 0,\ \text{as } k \to \infty. \tag{3.26}$$

Now we describe the method.

Step 0: Initialization

Set $X^0 \in \mathcal{K}$. Let k denote an iteration counter and set $k = 1$. Set the sequence a_k such that condition (3.26) is satisfied.

Step 1: Computation

Calculate $X^k \in \mathcal{K}$ solving the following variational inequality subproblem:

$$\langle X^k + a_k F(X^{k-1}) - X^{k-1}, X - X^k \rangle \geq 0, \quad \forall X \in \mathcal{K}.$$

Step 2: Convergence

Fix a tolerance $\epsilon > 0$ and check whether $|X^k - X^{k-1}| \leq \epsilon$, then stop; otherwise, set $k := k + 1$, and go to Step 1. The explicit formulas for the Euler method used in the transplant model are as follows:

$$X^k = \max\left\{0, X^{k-1} - a_{k-1} F(X^{k-1})\right\}.$$

3.5 Numerical Examples

In this section we present some numerical examples using the model described in Section 3.2.

According to SIT - Sistema Informativo Trapianti (Information Transplant System), in Italy in the last years the number of donors has greatly increased passing from 329 in 1992 to 1303 in 2016, as you can see in Figure 4.

Moreover, comparing the years 2015 and 2016, as in Table 1, in almost all cases the percentage increase has exceeded 10% and sometimes even 30%.

	Kidney	Liver	Hearth	Lung	Pancreas
2015	1580	1071	246	112	50
2016	1813	1215	267	154	69

Table 3.1: Number of transplants

According to the Italian DRG, which is the system that gives a cash value to the diagnosis and medical and surgical procedures, the process of an organ transplant, including withdrawal from the donor, transportation, storage and implantation into the recipient, have the following costs:

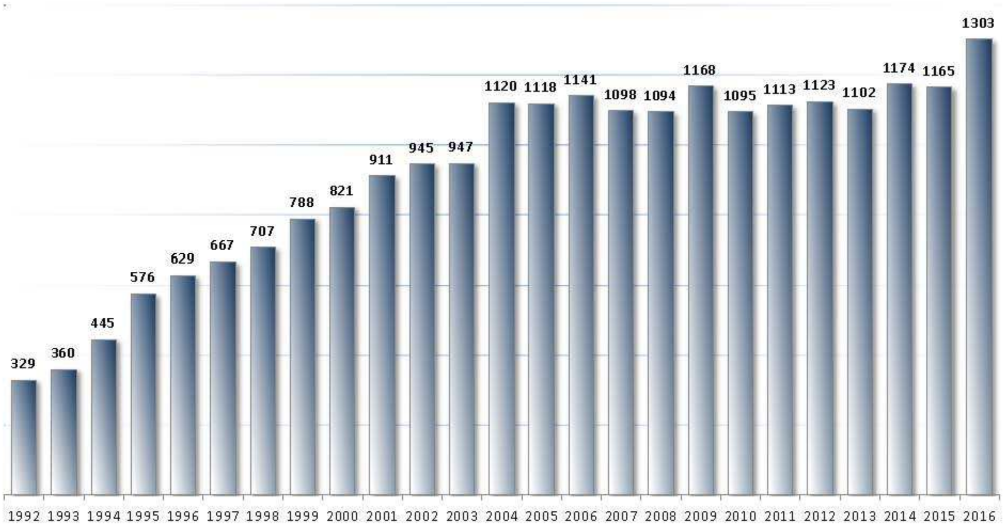

Figure 3.4: Number of used donors

	Kidney	Liver	Hearth	Lung	Pancreas
€	43,000	83,000	62,000	72,000	70,000

Table 3.2: Costs of organ transplants

In the following examples we consider some quadratic cost functions which in some sense represent the reality, since the marginal cost functions decrease when a large number of transplant is performed.

The costs are in thousands of euros and the organs are in hundreds of units.

The optimal solutions are calculated by applying the Euler method described in Section 3.4. The calculations were performed using the Matlab program. The algorithm was implemented on a laptop with an Intel Core2 Duo processor and 4 GB RAM. For the convergence of the method a tolerance $\epsilon = 10^{-4}$ was fixed. Specifically, the method has been implemented with a constant step $\alpha = 0.1$.

For all the analyzed cases, we have depicted the underlying network and specified the cost functions. We have also presented some small variations of examples 2 and 3 with equal or different link costs.

3.5.1 Example 1

In the first example we consider a simple network consisting of one transplant center, one donor hospital and only one transportation mode, as depicted in Figure 5.

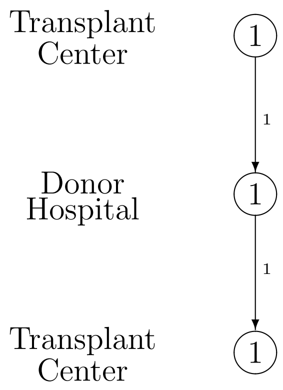

Figure 3.5: Organ transport network: Ex. 1

We assume the following cost functions are given:

$$\begin{aligned} c_{111}^{TE}(g_{111}) &= 0.25g_{111}^2 - 0.2g_{111} + 8, \\ c_1^S(g_1) &= 0.7g_1^2 - 0.5g_1 + 3, \\ c_{111}^{TO}(\tilde{g}_{111}) &= 0.75\tilde{g}_{111}^2 - 0.4\tilde{g}_{111} + 25, \\ \tilde{c}_1^S(\tilde{g}_1) &= 1.3\tilde{g}_1^2 - \tilde{g}_1 + 7, \\ c_1^{POST}(\tilde{g}_1) &= 1.5\tilde{g}_1^2 - 1.1\tilde{g}_1 + 17. \end{aligned}$$

Further, let the portion of explanted organs discarded in the donor hospital be $\beta_1 = 0.5$, the portion of organs reaching the transplant center, but which cannot be transplanted, be $\gamma_1 = 0.1$ and the unit special waste disposal costs be:

$$c_1^W = 3, \qquad \tilde{c}_1^W = 3.5.$$

Solving the associated variational inequality, we get the following optimal solution:

$$g_{111}^* = 0.80, \qquad g_1^* = 2.65, \qquad \tilde{g}_{111}^* = 1.32, \qquad \tilde{g}_1^* = 1.19.$$

3.5.2 Example 2

We now consider a network consisting of one transplant center, two donor hospitals and only one transportation mode, as depicted in Figure 6.

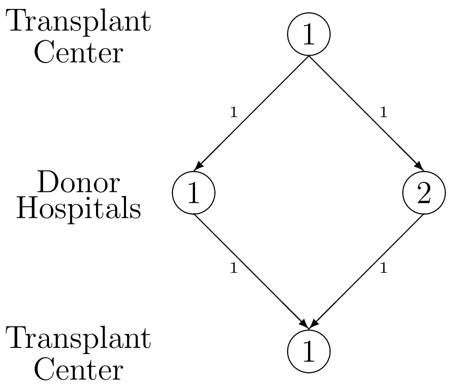

Figure 3.6: Organ transport network: Ex. 2

In this first case, we assume to have different link costs. Specifically, the following cost functions are given:

$$\begin{aligned}
c_{111}^{TE}(g_{111}) &= 0.3g_{111}^2 - 0.25g_{111} + 14,\\
c_{121}^{TE}(g_{121}) &= 0.4g_{121}^2 - 0.3g_{121} + 32,\\
c_1^S(g_1) &= 0.1g_1^2 - 0.09g_1 + 16,\\
c_1^S(g_2) &= 0.6g_2^2 - 0.5g_2 + 14,\\
c_{111}^{TO}(\tilde{g}_{111}) &= 0.2\tilde{g}_{111}^2 - 0.15\tilde{g}_{111} + 6,\\
c_{211}^{TO}(\tilde{g}_{211}) &= 0.7\tilde{g}_{211}^2 - 0.6\tilde{g}_{211} + 7,\\
\tilde{c}_1^S(\tilde{g}_1) &= 0.35\tilde{g}_1^2 - 0.3\tilde{g}_1 + 12,\\
c_1^{POST}(\tilde{g}_1) &= 0.6\tilde{g}_1^2 - 0.5\tilde{g}_1 + 16.
\end{aligned}$$

Further, let the portion of explanted organs discarded in the donor hospital 1 be $\beta_1 = 0.5$ and in the donor hospital 2 be $\beta_2 = 0.7$, the portion of organs reaching the transplant center, but which cannot be transplanted, be $\gamma_1 = 0.1$ and the unit special waste disposal costs be:

$$c_1^W = 1.5, \qquad c_2^W = 2.5, \qquad \tilde{c}_1^W = 1.7.$$

Solving the associated variational inequality, we get the following optimal solution:

$$\begin{aligned}
&g_{111}^* = 0.83, \quad g_{121}^* = 0.87, \quad g_1^* = 6.96, \quad g_2^* = 3.18,\\
&\tilde{g}_{111}^* = 3.48, \quad \tilde{g}_{211}^* = 0.95, \quad \tilde{g}_1^* = 3.99.
\end{aligned}$$

Keeping the same structure as the one depicted in Figure 5, we suppose now that the link costs are symmetrically the same. Specifically, we assume the following cost functions and the unit special waste disposal costs are given:

$$\begin{aligned}
c_{111}^{TE}(g_{111}) &= 0.3g_{111}^2 - 0.25g_{111} + 14,\\
c_{121}^{TE}(g_{121}) &= 0.3g_{121}^2 - 0.25g_{121} + 14,\\
c_1^S(g_1) &= 0.1g_1^2 - 0.09g_1 + 16,\\
c_2^S(g_2) &= 0.1g_2^2 - 0.09g_2 + 16,\\
c_{111}^{TO}(\tilde{g}_{111}) &= 0.2\tilde{g}_{111}^2 - 0.15\tilde{g}_{111} + 6,\\
c_{211}^{TO}(\tilde{g}_{211}) &= 0.2\tilde{g}_{211}^2 - 0.15\tilde{g}_{211} + 6,\\
\tilde{c}_1^S(\tilde{g}_1) &= 0.4\tilde{g}_1^2 - 0.3\tilde{g}_1 + 11,\\
c_1^{POST}(\tilde{g}_1) &= 0.7\tilde{g}_1^2 - 0.6\tilde{g}_1 + 14,\\
c_1^W &= 1.5,\\
c_2^W &= 1.5,\\
\tilde{c}_1^W &= 1.7.
\end{aligned}$$

In this case, we get the following optimal solution:

$$\begin{aligned}
&g_{111}^* = 0.83, \quad g_{121}^* = 0.83, \quad g_1^* = 5.32, \quad g_2^* = 5.75,\\
&\tilde{g}_{111}^* = 2.66, \quad \tilde{g}_{211}^* = 1.72, \quad \tilde{g}_1^* = 3.94.
\end{aligned}$$

The optimal solution of the second case clearly shows the symmetry of the network, since the same number of medical teams reaches the two donors hospitals, which offer the same number of organs. Comparing the two situations, we note that in the first case only one organ is lost during the transport, whereas in the second case only one half of the transported organs is transplanted.

3.5.3 Example 3

We now consider a network consisting of two transplant centers, two donor hospitals and only one transportation mode, as depicted in Figure 7.
In this first case, we assume to have different link costs. Specifically, the following cost functions are given:

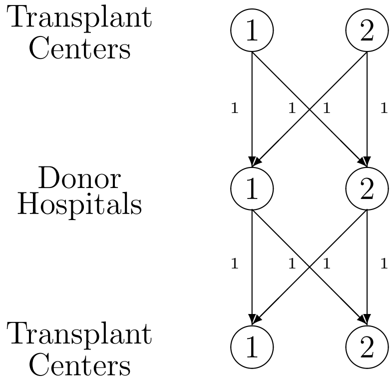

Figure 3.7: Organ transport network: Ex. 3

$$\begin{aligned}
c^{TE}_{111}(g_{111}) &= 0.25g^2_{111} - 0.2g_{111} + 17,\\
c^{TE}_{121}(g_{121}) &= 0.3g^2_{121} - 0.25g_{121} + 21,\\
c^{TE}_{211}(g_{211}) &= 0.35g^2_{211} - 0.3g_{211} + 26,\\
c^{TE}_{221}(g_{221}) &= 0.4g^2_{221} - 0.35g_{221} + 31,\\
c^{S}_{1}(g_{1}) &= 0.1g^2_{1} - 0.09g_{1} + 25,\\
c^{S}_{2}(g_{2}) &= 0.6g^2_{2} - 0.5g_{2} + 13,\\
c^{TO}_{111}(\tilde{g}_{111}) &= 0.2\tilde{g}^2_{111} - 0.15\tilde{g}_{111} + 6,\\
c^{TO}_{211}(\tilde{g}_{211}) &= 0.7\tilde{g}^2_{211} - 0.6\tilde{g}_{211} + 6,\\
c^{TO}_{121}(\tilde{g}_{121}) &= 1.2\tilde{g}^2_{121} - 1.1\tilde{g}_{121} + 2,\\
c^{TO}_{221}(\tilde{g}_{221}) &= 1.7\tilde{g}^2_{221} - 1.6\tilde{g}_{221} + 3,\\
\tilde{c}^{S}_{1}(\tilde{g}_{1}) &= 0.35\tilde{g}^2_{1} - 0.3\tilde{g}_{1} + 12,\\
\tilde{c}^{S}_{2}(\tilde{g}_{2}) &= 0.45\tilde{g}^2_{2} - 0.4\tilde{g}_{2} + 28,\\
c^{POST}_{1}(\tilde{g}_{1}) &= 0.6\tilde{g}^2_{1} - 0.5\tilde{g}_{1} + 16,\\
c^{POST}_{2}(\tilde{g}_{2}) &= 0.75\tilde{g}^2_{2} - 0.6\tilde{g}_{2} + 13.
\end{aligned}$$

Further, let the portion of explanted organs discarded in the donor hospital 1 be $\beta_1 = 0.5$ and in the donor hospital 2 be $\beta_2 = 0.7$, the portion of organs reaching the transplant centers, but which cannot be transplanted, be $\gamma_1 = 0.1$ and $\gamma_2 = 0.2$ respectively, and the unit special waste disposal costs be:

$$c_1^W = 1.5, \qquad c_2^W = 2.5,$$
$$\tilde{c}_1^W = 1.7, \qquad \tilde{c}_2^W = 1.9.$$

Solving the associated variational inequality, we get the following optimal solution:

$$\begin{array}{llll} g^*_{111} = 0.80, & g^*_{121} = 0.83, & g^*_{211} = 0.86, & g^*_{221} = 0.87, \\ g^*_1 = 10.89, & g^*_2 = 3.92, & \tilde{g}^*_{111} = 3.44 & \tilde{g}^*_{211} = 0.05, \\ \tilde{g}^*_{121} = 2.01, & \tilde{g}^*_{221} = 1.12, & \tilde{g}^*_1 = 3.14, & \tilde{g}^*_2 = 2.51. \end{array}$$

As in the case of Example 2, now we keep the same structure as the one depicted in Figure 6, but we suppose that the link costs are the same on some links. Specifically, we assume the following cost functions and the unit special waste disposal costs are given:

$$\begin{aligned}
c^{TE}_{111}(g_{111}) &= 0.25g^2_{111} - 0.2g_{111} + 17,\\
c^{TE}_{121}(g_{121}) &= 0.3g^2_{121} - 0.25g_{121} + 21,\\
c^{TE}_{211}(g_{211}) &= 0.25g^2_{211} - 0.2g_{211} + 17,\\
c^{TE}_{221}(g_{221}) &= 0.3g^2_{221} - 0.25g_{221} + 21,\\
c^{S}_{1}(g_{1}) &= 0.1g^2_{1} - 0.09g_{1} + 25,\\
c^{S}_{2}(g_{2}) &= 0.1g^2_{2} - 0.09g_{2} + 25,\\
c^{TO}_{111}(\tilde{g}_{111}) &= 0.2\tilde{g}^2_{111} - 0.15\tilde{g}_{111} + 6,\\
c^{TO}_{211}(\tilde{g}_{211}) &= 0.7\tilde{g}^2_{211} - 0.6\tilde{g}_{211} + 6,\\
c^{TO}_{121}(\tilde{g}_{121}) &= 0.2\tilde{g}^2_{121} - 0.15\tilde{g}_{121} + 6,\\
c^{TO}_{221}(\tilde{g}_{221}) &= 0.7\tilde{g}^2_{221} - 0.6\tilde{g}_{221} + 6,\\
\tilde{c}^{S}_{1}(\tilde{g}_{1}) &= 0.35\tilde{g}^2_{1} - 0.3\tilde{g}_{1} + 12,\\
\tilde{c}^{S}_{2}(\tilde{g}_{2}) &= 0.35\tilde{g}^2_{2} - 0.3\tilde{g}_{2} + 12,\\
c^{POST}_{1}(\tilde{g}_{1}) &= 0.6\tilde{g}^2_{1} - 0.5\tilde{g}_{1} + 16,\\
c^{POST}_{2}(\tilde{g}_{2}) &= 0.6\tilde{g}^2_{2} - 0.5\tilde{g}_{2} + 16,\\
c^{W}_{1} &= 1.5,\\
c^{W}_{2} &= 1.5,\\
\tilde{c}^{W}_{1} &= 1.7,\\
\tilde{c}^{W}_{2} &= 1.7.
\end{aligned}$$

In this case, we get the following optimal solution:

$$\begin{array}{llll}
g^*_{111} = 0.80, & g^*_{121} = 0.83, & g^*_{211} = 0.80, & g^*_{221} = 0.83,\\
g^*_{1} = 10.46, & g^*_{2} = 8.62, & \tilde{g}^*_{111} = 2.55 & \tilde{g}^*_{211} = 1.28,\\
\tilde{g}^*_{121} = 2.67, & \tilde{g}^*_{221} = 1.31, & \tilde{g}^*_{1} = 3.45, & \tilde{g}^*_{2} = 3.19.
\end{array}$$

In this version of Example 3, the structure of the network is still the one depicted in Figure 6, but the cost functions and the unit special waste disposal costs are now given by:

$$\begin{aligned}
c^{TE}_{111}(g_{111}) &= 0.3g^2_{111} - 0.25g_{111} + 21, \\
c^{TE}_{121}(g_{121}) &= 0.3g^2_{121} - 0.25g_{121} + 21, \\
c^{TE}_{211}(g_{211}) &= 0.4g^2_{211} - 0.3g_{211} + 31, \\
c^{TE}_{221}(g_{221}) &= 0.4g^2_{221} - 0.3g_{221} + 31, \\
c^{S}_{1}(g_{1}) &= 0.6g^2_{1} - 0.5g_{1} + 13, \\
c^{S}_{2}(g_{2}) &= 0.6g^2_{2} - 0.5g_{2} + 13, \\
c^{TO}_{111}(\tilde{g}_{111}) &= 0.7\tilde{g}^2_{111} - 0.6\tilde{g}_{111} + 6, \\
c^{TO}_{211}(\tilde{g}_{211}) &= 0.7\tilde{g}^2_{211} - 0.6\tilde{g}_{211} + 6, \\
c^{TO}_{121}(\tilde{g}_{121}) &= 1.7\tilde{g}^2_{121} - 1.6\tilde{g}_{121} + 3, \\
c^{TO}_{221}(\tilde{g}_{221}) &= 1.7\tilde{g}^2_{221} - 1.6\tilde{g}_{221} + 3, \\
\tilde{c}^{S}_{1}(\tilde{g}_{1}) &= 0.45\tilde{g}^2_{1} - 0.4\tilde{g}_{1} + 28, \\
\tilde{c}^{S}_{2}(\tilde{g}_{2}) &= 0.45\tilde{g}^2_{2} - 0.4\tilde{g}_{2} + 28, \\
c^{POST}_{1}(\tilde{g}_{1}) &= 0.75\tilde{g}^2_{1} - 0.7\tilde{g}_{1} + 13, \\
c^{POST}_{2}(\tilde{g}_{2}) &= 0.75\tilde{g}^2_{2} - 0.7\tilde{g}_{2} + 13, \\
c^{W}_{1} &= 2.5, \\
c^{W}_{2} &= 2.5, \\
\tilde{c}^{W}_{1} &= 1.9, \\
\tilde{c}^{W}_{2} &= 1.9.
\end{aligned}$$

Solving the associated variational inequality, we get the following optimal solution:

$$\begin{array}{llll}
g^*_{111} = 0.83, & g^*_{121} = 0.83, & g^*_{211} = 0.87, & g^*_{221} = 0.87, \\
g^*_{1} = 5.50, & g^*_{2} = 4.64, & \tilde{g}^*_{111} = 1.56 & \tilde{g}^*_{211} = 0.60, \\
\tilde{g}^*_{121} = 1.19, & \tilde{g}^*_{221} = 0.79, & \tilde{g}^*_{1} = 1.94, & \tilde{g}^*_{2} = 1.59.
\end{array}$$

By analyzing the optimal solutions of the three proposed cases, we observe that only in one case the number of medical teams is equal to the number of offered organs. In particular, the symmetry of the network in the second case, together with the proposed cost functions, allow us to obtain the greatest number of performed transplants. Furthermore, we observe that in the second case one organ is lost during the transport, whereas in the first and in the third cases all the transported organs are actually transplanted.

3.6 Conclusions

In the last decades the surgical techniques and medicines have undergone an extraordinary improvement, so that the number of organ transplants has reduced, but the demand is still much higher than the supply and the costs associated with the transplant process, including hospital and surgery costs, medical teams and organs transportation costs, and disposal costs are very expensive for the National Health Service. In this paper, we present a model which captures this significant issue and provide a variational formulation which characterizes the optimality conditions consisting in the minimization of the total costs. Existing results for the solution to the variational inequality are also stated.

The application of the Lagrange theory to the transplant model allows us to explain the meaning of some Lagrange multipliers in order to better understand the behavior of the entire process.

The theoretical framework is then further illustrated through some numerical examples for which the equilibrium amount of medical teams, of organs offered by the donor hospitals, of organs transported from the donor hospitals to the transplant centers, and of organs actually transplanted are computed. The results in this paper add to the existing literature of operations research techniques for transplant modeling and analysis.

Chapter 4

A Variational Equilibrium Network Framework for Humanitarian Organizations in Disaster Relief: Effective Product Delivery Under Competition for Financial Funds

In this chapter, we present a new Generalized Nash Equilibrium (GNE) model for post-disaster humanitarian relief by introducing novel financial funding functions and altruism functions, and by also capturing competition on the logistics side among humanitarian organizations. The common, that is, the shared, constraints associated with the relief item deliveries at points of need are imposed by an upper level humanitarian organization or regulatory body and consist of lower and upper bounds to ensure the effective delivery of the estimated volumes of supplies to the victims of the disaster. We identify the network structure of the problem, with logistical and financial flows, and propose a variational equilibrium framework, which allows us to then formulate, analyze, and solve the model using the theory of variational inequalities (rather than quasivariational inequality theory). We then utilize Lagrange analysis and investigate qualitatively the humanitarian organizations' marginal utilities if and when the equilibrium relief item flows are (or are not) at the imposed demand point bounds. We illustrate the game theory model through a case study focused on tornadoes hitting western Massachusetts, a highly unusual event that occurred in 2011. This work significantly extends the original model of Nagurney, Alvarez Flores, and Soylu (2016), which, under the imposed assumptions therein, allowed for an optimization formulation, and adds to the literature of game theory and disaster relief, which is nascent.

4.1 Introduction

Disaster relief is fraught with many challenges: the infrastructure, from transportation to communications to energy delivery, may be damaged or destroyed, and services, from healthcare to governmental ones, impacted, all while victims are in desperate need of relief items such as water, food, medicines, and shelter. A timely response to a disaster, hence, can save lives, reduce suffering, and assist in recovery. Moreover, it can also enhance the reputations of humanitarian organizations and their very sustainability in terms of financial donations.

The number of disasters is growing as well as the number of people affected by them (Nagurney and Qiang (2009)) with additional pressures coming from climate change, increasing growth of populations in urban environments, and the spread of diseases brought about by global air travel. The associated costs of the damage and losses due to disasters is estimated at an average \$100 billion a year since the turn of the century (Watson et al. (2015)).

Disasters come in many forms, from natural disasters, such as tornadoes, earthquakes, and typhoons, which are often sudden-onset, to famines, which are slow-onset, and can occur not only from changes in weather patterns, resulting in droughts, for example, but also from political situations, including war (cf. Van Wassenhove (2006)). Hence, certain disasters are man-made, as in the case of the Syrian refugee crisis (cf. Sumpf, Isaila, and Najjar (2016)), and terrorist attacks, such as 9/11 (Cox (2008)).

Notable sudden-onset natural disasters have included Hurricane Katrina in 2005, which was the costliest natural disaster in the US, the Haiti earthquake in 2010, the triple disaster in Fukushima, Japan in 2011, consisting of an earthquake, followed by a tsunami and a nuclear meltdown technological disaster, Superstorm Sandy in 2012, tropical cyclone Haiyan in 2013, which was the strongest cyclone ever recorded, the earthquake in Nepal in 2015, and Hurricane Matthew in 2016.

The challenges to disaster relief (humanitarian) organizations, including nongovernmental organizations (NGOs), are immense. The majority operate under a single, common, humanitarian principle of protecting the vulnerable, reducing suffering, and supporting the quality of life, while, at the same time, competing for financial funds from donors to ensure their own sustainability. As noted in Nagurney, Alvarez Flores, and Soylu (2016), competition is intense, with the number of registered US nonprofit organizations increasing from 12,000 in 1940 to more than 1.5 million in 2012. Approximately $300 billion are donated to charities in the United States each year (Zhuang, Saxton, and Wu (2014)). At the same time, many stakeholders believe that humanitarian aid has not been as successful in delivering on the humanitarian principle as might be feasible due to a lack of coordination, which results in duplication of services (see Kopinak (2013)).

We believe that some of the challenges that humanitarian organizations engaged in disaster relief are faced with can be addressed through the use of game theory. Game theory is a methodological framework that captures complex interactions among competing decision-makers (noncooperative games) or cooperating ones (cooperative games). The contributions of John Nash (1950, 1951), in particular, are highly relevant and established some of the foundations of game theory. Specifically, we note that, in the case of non-cooperative games, in which the utilities of the competing players, that is, the decision-makers, in the game, depend on the other players' strategies, the governing concept is that of Nash Equilibrium. If, however, the feasible sets, that is, the constraints, are not specific to each player, but, rather, depend also on the strategies of the other players, then we are dealing with a Generalized Nash Equilibrium, introduced by Debreu (1952) (see, also, von Heusinger (2009), Fischer, Herrich, and Schonefeld (2014), and the references therein).

In particular, in this paper, we construct a new Generalized Nash Equilibrium (GNE) network model for disaster relief, which models competition among NGOs for financial funds post-disaster, as well as for the delivery of relief items. The utility function that each NGO seeks to maximize depends on its financial gain from donations plus the weighted benefit accrued from doing good through the delivery of relief items minus the total cost associated with the logistics of delivering the relief items. The model extends the earlier model of Nagurney, Alvarez Flores, and Soylu (2016) in the following significant ways, which means that the optimization reformulation, as done in that paper, no longer applies:

1. The financial funds functions, which capture the amount of donations to each NGO, given their visibility through media of the supplies of relief items delivered at demand points, and under competition, need not take on a particular structure.
2. The altruism or benefit functions, also included in each NGO's utility function, need not be linear.
3. The competition associated with logistics is captured through total cost functions that depend not only on a particular NGO's relief item shipments but also on those of the other NGOs.

In order to guarantee effective product delivery at the demand points, we retain the lower and upper bounds, as introduced in Nagurney, Alvarez Flores, and Soylu (2016). Such common, or shared constraints, assist in coordination (cf. Balcik et al. (2010)) and would be imposed by a higher level humanitarian organization or regulatory authority in order to ensure that the needed volumes of relief items are delivered but are not oversupplied, which can results in congestion, materiel convergence, and wastage. We assume that the NGOs have prepositioned the supplies of the disaster relief items and that the total amount available across all NGOs is sufficient to meet the needs of the victims.

It is important to emphasize that Generalized Nash Equilibrium problems are more challenging to formulate and solve and are usually tackled via quasivariational inequalities (cf. Bensoussan (1974)), the theory of which, as well as the associated computational procedures, are not in as an advanced state as that of variational inequalities (see Kinderlehrer and Stampacchia (1980) and Nagurney (1999)). Here we utilize, for the first time, in the context of humanitarian operations and disaster relief, a *variational equilibrium*. As noted in Nagurney, Yu, and Besik (2017)), a variational equilibrium is a specific kind of GNE (cf. Facchinei and Kanzow (2010), Kulkarni and Shanbhag (2012)). The variational equilibrium allows for alternative variational inequality formulations of our new Generalized Nash Equilibrium network model. What is notable about a variational equilibrium (see also Luna (2013)) is that the Lagrange multipliers associated with the shared or coupling constraints of the NGOs are the same for all NGOs in the disaster relief game. This also provides us with an elegant economic and equity interpretation.

The only other game theory model for disaster relief that includes elements of logistics plus financial funds is that of Nagurney, Alvarez Flores, and Soylu (2016). Zhuang, Saxton, and Wu (2014) proposed a model that showed that the amount of charitable contributions made by donors is positively dependent on the amount of disclosure by the NGOs. The authors emphasized that there is a dearth of existing game-theoretic research on nonprofit organizations. Toyasaki and Wakolbinger (2015) developed game theory models to analyze whether an NGO should establish a special fund after a disaster (in terms of earmarked donations) or allow only unearmarked donations. Nagurney (2016), in turn, presented a network game theory model in which multiple freight service providers are engaged in competition to acquire the business of carrying disaster relief supplies of a humanitarian organization in the amounts desired to the destinations. Coles and Zhuang (2011), on the other hand, argued for the need for cooperative game theory models for disaster recovery operations by highlighting a stream of post-disaster operations. Muggy and Stamm (2014) give an excellent review of game theory in humanitarian operations and note that there are many untapped research opportunities for modeling in this area. See also the dissertation of Muggy (2015). The research in our paper adds to the still nascent literature on game theory and disaster relief / humanitarian operations.

This paper is organized as follows. In Section 2, we construct the novel Generalized Nash Equilibrium model for disaster relief, which captures competition both on the financial funds side as well as on the logistics side and we identify the network structure. We present the variational equilibrium framework and also prove the existence of an equilibrium solution. In addition, we provide, for completeness, the variational inequality formulation of a special case of the model, under the Nash equilibrium solution, in the absence of imposed common demand constraints. In Section 3, we then explore, through Lagrange analysis, the humanitarian organizations' marginal utilities when the equilibrium disaster relief flows are at the upper or the lower bounds of the imposed demands of the regulatory body or lie in between. In order to illustrate the framework developed here, Section 4 presents both an algorithmic scheme and a case study, inspired by tornadoes that hit western Massachusetts in June 2011, with devastating impact (cf. Western Massachusetts Regional Homeland Advisory Council (2012)). We summarize our results and present our conclusions in Section 5.

4.2 The Variational Equilibrium Network Framework for Humanitarian Organizations in Disaster Relief

We now present the new Generalized Nash Equilibrium model for disaster relief, along with the variational equilibrium framework. As mentioned in the Introduction, the model extends the earlier model of Nagurney, Alvarez Flores, and Soylu (2016), which, under the imposed assumptions therein, allowed for an optimization reformulation. Our notation follows closely the notation in the above paper but here we utilize, in contrast, a more general variational equilibrium framework.

We consider m humanitarian organizations, here referred to as nongovernmental organizations (NGOs), with a typical NGO denoted by i, seeking to deliver relief supplies, post a disaster, to n demand points, with a typical demand point denoted by j. The relief supplies can be water, food, or medicine. We assume that the product delivered can be viewed as being homogeneous. We denote the volume of the relief item shipment (flow) delivered by NGO i to demand point j by q_{ij}. We group the nonnegative relief item flows from each NGO i; $i = 1, \ldots, m$, into the vector $q_i \in R^n_+$ and then we group the relief item flows of all the NGOs to all the demand points into the vector $q \in R^{mn}_+$. The vector q_i is the vector of strategies of NGO i.

The NGOs compete for financial funds from donors, while also engaging in competition on the logistic side in terms of costs, since there may be competition for freight services, etc., as well as congestion at the demand sites. The network structure of the problem is given in Figure 1. Note that the links from the first tier nodes representing the NGOs to the bottom tier nodes, corresponding to the demand points, are the shipment links and have relief item flows associated with them. The links from the demand nodes to the NGO nodes (in the opposite direction) are the links with the financial flows from the donors reacting to the visibility of the NGOs in their delivery of the needed supplies through the media. The network structure of this problem differs from the network underlying the model given in Nagurney, Alvarez Flores, and Soylu (2016) since in that model, the financial flows, once collected, were partitioned to each NGO, using a factor representing the portion of the financial funds each humanitarian organization was (likely) to get of the total amount donated.

We emphasize that, in terms of the sequence of events, the humanitarian organizations first decide on the level of relief items to be provided at each demand point and deliver the amounts. Then they receive the corresponding financial flows. Therefore, the financial flows are received after the supplies arrive. As noted in Nagurney, Alvarez Flores, and Soylu (2016), empirically, these funds are realized and made available quickly, and these two events are almost concurrent in many cases. The justification of this assumption is also provided by the nature of the incentives of the decision-makers in our model, which is to provide humanitarian relief as quickly as possible whenever a disaster strikes.

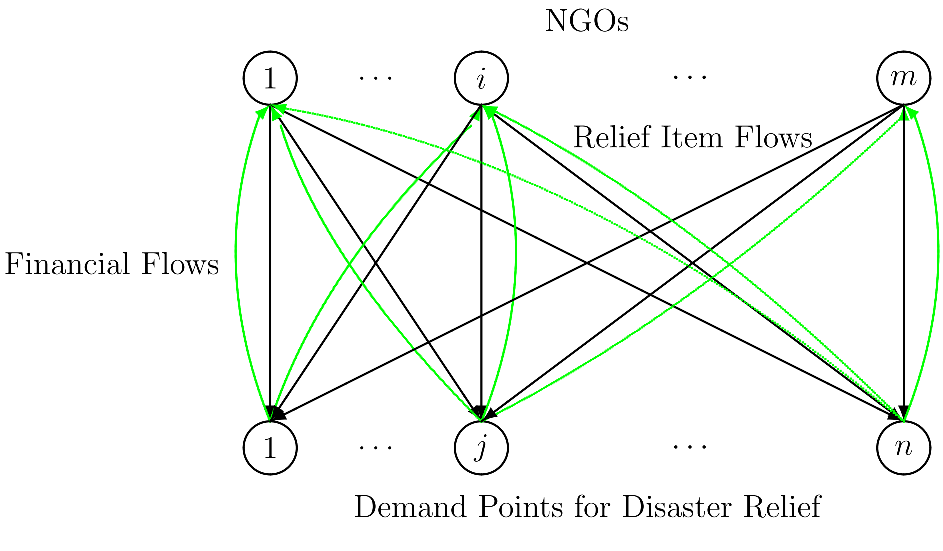

Figure 4.1: The Network Structure of the Game Theory Model

Each NGO i encumbers a cost, c_{ij}, associated with shipping the relief items to location j, where we assume that

$$c_{ij} = c_{ij}(q), \quad j = 1, \ldots n, \tag{4.1}$$

with these cost functions being convex and continuously differentiable. These costs also include transaction costs (see also Nagurney (2006)). Note that the cost functions (4.1) are associated with the logistics aspects and, hence, the cost on a shipment link can depend not only on its flow but also on the flows on the other shipment links associated with the same NGO or with other NGOs.

Each NGO i; $i = 1, \ldots, m$, based on the media attention and the visibility of NGOs at demand point j; $j = 1, \ldots, n$, receives financial funds from donors given by the expression

$$\sum_{j=1}^{n} P_{ij}(q), \tag{4.2}$$

where $P_{ij}(q)$ denotes the financial funds in donation dollars given to NGO i due to visibility of NGO i at location j. Hence, $P_{ij}(q)$ corresponds to the financial flow on the link joining demand node j with node NGO i in Figure 1. Observe that, according to (4.2), there is competition among all the NGOs for financial donations since the financial amount of donations that an NGO receives depends not only on its relief item deliveries but also on those delivered by other NGOs. Indeed, according to (4.2), an NGO may benefit from donations even through visibility of other NGOs providing the product because of, for example, loyalty and support for a specific NGO. We assume that the P_{ij} functions are increasing, concave, and continuously differentiable. Hence, we have positive but decreasing marginal utility of providing aid (in terms of the NGO's effect on attracting donations). It is important to mention that the $P_{ij}(q)$ function contains, as a special case, the financial funds donor function of Nagurney, Alvarez Flores, and Soylu (2016), with $P_{ij}(q) = \beta_i P_j(q)$; $i = 1, \ldots, m$; $j = 1, \ldots, n$. Furthermore, Natsios (1995) noted that the cheapest way for relief organizations to fundraise is to provide early relief in highly visible areas. In our case study in Section 4 we construct explicit $P_{ij}(q)$ functions for all NGOs i and demand points j.

Also, since the NGOs are humanitarian organizations involved in disaster relief, each NGO i also derives some utility from delivering the needed relief supplies. We, hence, introduce an altruism/benefit function B_i; $i = 1, \ldots, m$, such that

$$B_i = B_i(q), \tag{4.3}$$

and each benefit function is assumed to be concave and continuously differentiable. Previously utilized benefit functions in this application domain were of the form: $B_i = \sum_{j=1}^{n} \gamma_{ij} q_{ij}$; $j = 1, \ldots, n$. Furthermore, when we construct each NGO's full utility function we will also assign a weight ω_i before each $B_i(q)$; $i = 1, \ldots, m$, to represent a monetized weight associated with altruism of i. Such weight concepts are used in multicriteria decision-making; see, e.g., Fishburn (1970), Chankong and Haimes (1983), Yu (1985), Keeney and Raiffa (1993), and Nagurney, Alvarez Flores, and Soylu (2016).

Each NGO i; $i = 1, \ldots, m$, has an amount s_i of the relief item that it can allocate post-disaster, which must satisfy:

$$\sum_{j=1}^{n} q_{ij} \leq s_i. \tag{4.4}$$

We assume that the relief supplies have been prepositioned so that they are in stock and available, since time is of the essence. According to Roopanarine (2013), prepositioning of supplies can make emergency relief more effective and this is a strategy followed not only by the UNHRD (United Nations Humanitarian Response Depot) but also by the Red Cross and even some smaller relief organizations such as AmeriCares. Gatignon, Van Wassenhove, and Charles (2010) also note the benefits of proper prepositioning of supplies in the case of the International Federation of the Red Cross (IFRC) in terms of cost reduction and a more timely response.

In addition, the relief item flows for each i; $i = 1, \ldots, m$, must be nonnegative, that is:

$$q_{ij} \geq 0, \quad j = 1, \ldots, n. \tag{4.5}$$

Each NGO i; $i = 1, \ldots, m$, seeks to maximize its utility, U_i, with the utility consisting of the financial gains due to its visibility through media of the relief item flows, $\sum_{j=1}^{n} P_{ij}(q)$, plus the utility associated with the logistical (supply chain) aspects of delivery of the supplies, which consists of the weighted altruism/benefit function minus the logistical costs. For additional background on utility functions for nonprofit and charitable organizations, see Rose-Ackerman (1982) and Malani, Philipson, and David (2003).

Without the imposition of demand bound constraints (which will follow), the optimization problem faced by NGO i; $i = 1, \ldots, m$, is, thus,

$$\text{Maximize} \quad U_i(q) = \sum_{j=1}^{n} P_{ij}(q) + \omega_i B_i(q) - \sum_{j=1}^{n} c_{ij}(q) \tag{4.6}$$

subject to constraints (4.4) and (4.5).

Before imposing the common constraints, we remark that the above model, in the absence of any common constraints, is a Nash Equilibrium problem, which we know can be formulated and solved as a variational inequality problem (cf. Gabay and Moulin (1980) and Nagurney (1999)). Indeed, although the utility functions of the NGOs depend on their strategies and those of the other NGOs, the respective NGO feasible sets do not. However, the NGOs may be faced with several common constraints, which make the game theory problem more complex and challenging. The common constraints, which are imposed by an authority, such as a governmental one or a higher level humanitarian coordination agency, ensure that the needs of the disaster victims are met, while recognizing the negative effects of waste and material convergence. The imposition of such constraints in terms of effectiveness and even gains for NGOs was demonstrated in Nagurney, Alvarez Flores, and Soylu (2016). Later in this section, we present the variational inequality framework. Hence, we will not need to make use of quasivariational inequalities (cf. von Heusinger (2009)) for our new model.

Specifically, the two sets of common imposed constraints, at each demand point j; $j = 1, \ldots, n$, are as follows:

$$\sum_{i=1}^{m} q_{ij} \geq \underline{d}_j, \tag{4.7}$$

and

$$\sum_{i=1}^{m} q_{ij} \leq \bar{d}_j, \tag{4.8}$$

where $\underline{d}_j$ is the lower bound on the amount of the relief item needed at demand point j and $\bar{d}_j$ is the upper bound on the amount of the relief item needed at demand point j. The constraints (4.7) and (4.8) give flexibility for a regulatory or coordinating body, since it is not likely that the demand will be precisely known in a disaster situation. It is, however, reasonable to assume that, as represented in these equations, estimates for needs assessment for the relief items will be available at the local level.

We assume that

$$\sum_{i=1}^{m} s_i \geq \sum_{j=1}^{n} \underline{d}_j. \tag{4.9}$$

Hence, the total supply of the relief item of the NGOs is sufficient to meet the needs at all the demand points.

We define the feasible set K_i for each NGO i as:

$$K_i \equiv \{q_i | \, (4.4) \, \text{and} \, (4.5) \, \text{hold}\} \tag{4.10}$$

and we let $K \equiv \prod_{i=1}^{m} K_i$.

In addition, we define the feasible set $\mathcal{S}$ consisting of the shared constraints as:

$$\mathcal{S} \equiv \{q | \, (4.7) \, \text{and} \, (4.8) \, \text{hold}\}. \tag{4.11}$$

Observe that now not only does the utility of each NGO depend on the strategies, that is, the relief item flows, of the other NGOs, but so does the feasible set because of the common constraints (4.7) and (4.8). Hence, the above game theory model, in which the NGOs compete noncooperatively is a Generalized Nash Equilibrium problem. Therefore, we have the following definition.

Definition 1: Disaster Relief Generalized Nash Equilibrium

A relief item flow pattern $q^* \in K = \prod_{i=1}^{m} K_i$, $q^* \in \mathcal{S}$, *constitutes a disaster relief Generalized Nash Equilibrium if for each NGO* i; $i = 1, \ldots, m$:

$$\hat{U}_i(q_i^*, \hat{q}_i^*) \geq U_i(q_i, \hat{q}_i^*), \quad \forall q_i \in K_i, \forall q \in \mathcal{S}, \tag{4.12}$$

where $\hat{q}_i^* \equiv (q_1^*, \ldots, q_{i-1}^*, q_{i+1}^*, \ldots, q_m^*)$.

Hence, an equilibrium is established if no NGO can unilaterally improve upon its utility by changing its relief item flows in the disaster relief network, given the relief item flow decisions of the other NGOs, and subject to the supply constraints, the nonnegativity constraints, and the shared/coupling constraints. We remark that both K and $\mathcal{S}$ are convex sets.

If there are no coupling, that is, shared, constraints in the above model, then q and q^* in Definition 1 need only lie in the set K, and, under the assumption of concavity of the utility functions and that they are continuously differentiable, we know that (cf. Gabay and Moulin (1980) and Nagurney (1999)) the solution to what would then be a Nash equilibrium problem (see Nash (1950, 1951)) would coincide with the solution of the following variational inequality problem: determine $q^* \in K$, such that

$$-\sum_{i=1}^{m} \langle \nabla_{q_i} \hat{U}_i(q^*), q_i - q_i^* \rangle \geq 0, \quad \forall q \in K, \tag{4.13}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in the corresponding Euclidean space and $\nabla_{q_i} \hat{U}_i(q)$ denotes the gradient of $\hat{U}_i(q)$ with respect to q_i.

As emphasized in Nagurney, Yu, and Besik (2017), a refinement of the Generalized Nash Equilibrium is what is known as a variational equilibrium and it is a specific type of GNE (see Kulkarni and Shabhang (2012)). Specifically, in a GNE defined by a variational equilibrium, the Lagrange multipliers associated with the common/shared/coupling constraints are all the same. This feature provides a fairness interpretation and is reasonable from an economic standpoint. More precisely, we have the following definition:

Definition 2: Variational Equilibrium

A strategy vector q^* is said to be a variational equilibrium of the above Generalized Nash Equilibrium game if $q^* \in K, q^* \in \mathcal{S}$ is a solution of the variational inequality:

$$-\sum_{i=1}^{m} \langle \nabla_{q_i} U_i(q^*), q_i - q_i^* \rangle \geq 0, \quad \forall q \in K, \forall q \in \mathcal{S}. \tag{4.14}$$

By utilizing a variational equilibrium, we can take advantage of the well-developed theory of variational inequalities, including algorithms (cf. Nagurney (1999) and the references therein), which are in a more advanced state of development and application than algorithms for quasivariational inequality problems.

We now expand the terms in variational inequality (4.14).

Specifically, we have that (4.14) is equivalent to the variational inequality: determine $q^* \in K, q^* \in \mathcal{S}$, such that

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \left[\sum_{k=1}^{n} \frac{\partial c_{ik}(q^*)}{\partial q_{ij}} - \sum_{k=1}^{n} \frac{\partial P_{ik}(q^*)}{\partial q_{ij}} - \omega_i \frac{\partial B_i(q^*)}{\partial q_{ij}} \right] \times \left[q_{ij} - q_{ij}^* \right] \geq 0, \quad \forall q \in K, \forall q \in \mathcal{S}. \tag{4.15}$$

4.3 Lagrange Theory and Analysis of the Marginal Utilities

In this section we explore the Lagrange theory associated with variational inequality (4.15) and we provide an analysis of the marginal utilities at the equilibrium solution. For an application of Lagrange theory to other models, see: Daniele (2001) (spatial economic models), Barbagallo, Daniele, and Maugeri (2012) (financial networks), Toyasaki, Daniele, and Wakolbinger (2014) (end-of-life products networks), Daniele and Giuffrè (2015) (random traffic networks), Caruso and Daniele (2016) (transplant networks), Nagurney and Dutta (2016) (competition for blood donations).

By setting:

$$C(q)=\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\sum_{k=1}^{n}\frac{\partial c_{ik}(q^*)}{\partial q_{ij}}-\sum_{k=1}^{n}\frac{\partial P_{ik}(q^*)}{\partial q_{ij}}-\omega_i\frac{\partial B_i(q^*)}{\partial q_{ij}}\right](q_{ij}-q_{ij}^*), \quad (4.16)$$

variational inequality (4.15) can be rewritten as a minimization problem as follows:

$$\min_{\mathcal{K}} C(q)=C(q^*)=0. \quad (4.17)$$

Under the previously imposed assumptions, we know that all the involved functions in (4.17) are continuously differentiable and convex.

We set:

$$\begin{aligned} a_{ij} &= -q_{ij}\le 0, && \forall i,\ \forall j,\\ b_i &= \sum_{j=1}^{n} q_{ij}-s_i\le 0, && \forall i,\\ c_j &= \underline{d}_j-\sum_{i=1}^{m} q_{ij}\le 0, && \forall j,\\ e_j &= \sum_{i=1}^{m} q_{ij}-\overline{d}_j\le 0, && \forall j, \end{aligned} \quad (4.18)$$

and

$$\Gamma(q)=(a_{ij},b_i,c_j,e_j)_{i=1,\dots,m;\,j=1,\dots,n}\,. \quad (4.19)$$

As a consequence, we remark that $\mathcal{K}$ can be rewritten as

$$\mathcal{K}=\{q\in R^{mn}:\Gamma(q)\le 0\}. \quad (4.20)$$

We now consider the following Lagrange function:

$$\begin{aligned} \mathcal{L}(q,\alpha,\delta,\sigma,\varepsilon) &= \sum_{j=1}^{n} c_{ij}(q)-\sum_{j=1}^{n} P_{ij}(q)-\omega_i B_i(q)\\ &+ \sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_{ij}a_{ij}+\sum_{i=1}^{m}\delta_i b_i+\sum_{j=1}^{n}\sigma_j c_j+\sum_{j=1}^{n}\varepsilon_j e_j, \end{aligned} \quad (4.21)$$

$$\forall q\in R^{mn}_+,\ \forall\alpha\in R^{mn}_+,\ \forall\delta\in R^{m}_+,\ \forall\sigma\in R^{n}_+,\ \forall\varepsilon\in R^{n}_+,$$

where α is the vector with components: $\{\alpha_{11},\dots,\alpha_{mn}\}$; δ is the vector with components $\{\delta_1,\dots,\delta_m\}$; σ is the vector with elements: $\{\sigma_1,\dots,\sigma_n\}$, and ϵ is the vector with elements: $\{\epsilon_1,\dots,\epsilon_n\}$.

It is easy to prove that the feasible set $\mathcal{K}$ is convex and that the Slater condition is satisfied. Then, if q^* is a minimal solution to problem (4.17), there exist $\alpha^* \in R^{mn}_+$, $\delta^* \in R^m_+$, $\sigma^* \in R^n_+$, $\varepsilon^* \in R^n_+$ such that the vector $(q^*, \alpha^*, \delta^*, \sigma^*, \varepsilon^*)$ is a saddle point of the Lagrange function (4.21); namely:

$$\mathcal{L}(q^*, \alpha, \delta, \sigma, \varepsilon) \le \mathcal{L}(q^*, \alpha^*, \delta^*, \sigma^*, \varepsilon^*) \le \mathcal{L}(q, \alpha^*, \delta^*, \sigma^*, \varepsilon^*), \tag{4.22}$$

$$\forall q \in R^{mn}_+, \ \forall \alpha \in R^{mn}_+, \ \forall \delta \in R^m_+, \ \forall \sigma \in R^n_+, \ \forall \varepsilon \in R^n_+,$$

and

$$\alpha^*_{ij} a^*_{ij} = 0, \ \forall i, \ \forall j,$$

$$\delta^*_i b^*_i = 0, \ \forall i,$$

$$\sigma^*_j c^*_j = 0, \quad \varepsilon^*_j e^*_j = 0, \ \forall j. \tag{4.23}$$

From the right-hand side of (4.22), it follows that $q^* \in R^{mn}_+$ is a minimal point of $\mathcal{L}(q, \alpha^*, \delta^*, \sigma^*, \varepsilon^*)$ in the whole space R^{mn}, and hence, for all $i = 1, \ldots, m$, and for all $j = 1, \ldots, n$, we have that:

$$\frac{\partial \mathcal{L}(q^*, \alpha^*, \delta^*, \sigma^*, \varepsilon^*)}{\partial q_{ij}}$$

$$= \sum_{k=1}^{n} \frac{\partial c_{ik}(q^*)}{\partial q_{ij}} - \sum_{k=1}^{n} \frac{\partial P_{ik}(q^*)}{\partial q_{ij}} - \omega_i \frac{\partial B_i(q^*)}{\partial q_{ij}} - \alpha^*_{ij} + \delta^*_i - \sigma^*_j + \varepsilon^*_j = 0, \tag{4.24}$$

together with conditions (4.23).

Conditions (4.23) and (4.24) represent an equivalent formulation of variational inequality (4.15). Indeed, if we multiply (4.24) by $(q_{ij} - q^*_{ij})$ and sum up with respect to i and j, we get:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \left[\sum_{k=1}^{n} \frac{\partial c_{ik}(q^*)}{\partial q_{ij}} - \sum_{k=1}^{n} \frac{\partial P_{ik}(q^*)}{\partial q_{ij}} - \omega_i \frac{\partial B_i(q^*)}{\partial q_{ij}} \right] (q_{ij} - q^*_{ij})$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha^*_{ij} q_{ij} - \underbrace{\sum_{i=1}^{m} \sum_{j=1}^{n} \alpha^*_{ij} q^*_{ij}}_{=0} - \sum_{i=1}^{m} \left(\delta^*_i \sum_{j=1}^{n} q_{ij} - \underbrace{\delta^*_i \sum_{j=1}^{n} q^*_{ij}}_{=\delta^*_i s_i} \right)$$

$$+\sum_{j=1}^{n}\left(\sigma_j^*\sum_{i=1}^{m}q_{ij}-\underbrace{\sigma_j^*\sum_{i=1}^{m}q_{ij}^*}_{=\sigma_j^*\underline{d}_j}\right)-\sum_{j=1}^{n}\left(\varepsilon_j^*\sum_{i=1}^{m}q_{ij}-\underbrace{\varepsilon_j^*\sum_{i=1}^{m}q_{ij}^*}_{=\varepsilon_j^*\overline{d}_j}\right)$$

$$=\sum_{i=1}^{m}\sum_{j=1}^{n}\underbrace{\alpha_{ij}^*q_{ij}}_{\geq 0}-\sum_{i=1}^{m}\delta_i^*\left(\underbrace{\sum_{j=1}^{n}q_{ij}-s_i}_{\leq 0}\right)+\sum_{j=1}^{n}\sigma_j^*\left(\underbrace{\sum_{i=1}^{m}q_{ij}-\underline{d}_j}_{\geq 0}\right)$$

$$-\sum_{j=1}^{n}\varepsilon_j^*\left(\underbrace{\sum_{i=1}^{m}q_{ij}-\overline{d}_j}_{\leq 0}\right)\geq 0. \qquad (4.25)$$

We now discuss the meaning of some of the Lagrange multipliers. We focus on the case where $q_{ij}^*>0$; namely, the relief item flow from NGO i to demand point j is positive; otherwise, if $q_{ij}^*=0$, the problem is not interesting. Then, from the first line in (4.23), we have that $\alpha_{ij}^*=0$.

Let us consider the situation when the constraints are not active, that is, $b_i^*<0$ and $\underline{d}_j<\sum_{i=1}^{m}q_{ij}^*<\overline{d}_j$.

Specifically, $b_i^*<0$ means that $\sum_{j=1}^{n}q_{ij}^*<s_i$; that is, the sum of relief items sent by the i-th NGO to all demand points is strictly less than the total amount s_i at its disposal. Then, from the second line in (4.23), we get: $\delta_i^*=0$.

At the same time, from the last line in (4.23), $\underline{d}_j<\sum_{i=1}^{m}q_{ij}^*<\overline{d}_j$, leads to: $\sigma_j^*=\varepsilon_j^*=0$.

Hence, (4.24) yields:

$$\sum_{k=1}^{n}\frac{\partial c_{ik}(q^*)}{\partial q_{ij}}-\sum_{k=1}^{n}\frac{\partial P_{ik}(q^*)}{\partial q_{ij}}-\omega_i\frac{\partial B_i(q^*)}{\partial q_{ij}}=\alpha_{ij}^*-\delta_i^*+\sigma_j^*-\varepsilon_j^*=0$$

$$\Longleftrightarrow \sum_{k=1}^{n}\frac{\partial P_{ik}(q^*)}{\partial q_{ij}}+\omega_i\frac{\partial B_i(q^*)}{\partial q_{ij}}=\sum_{k=1}^{n}\frac{\partial c_{ik}(q^*)}{\partial q_{ij}}. \qquad (4.26)$$

In this case, the marginal utility associated with the financial donations plus altruism is equal to the marginal costs.

If, on the other hand, $\sum_{i=1}^{m} q_{ij}^* = \underline{d}_j$, then $\sigma_j^* > 0$. Hence, we get:

$$\sum_{k=1}^{n} \frac{\partial P_{ik}(q^*)}{\partial q_{ij}} + \omega_i \frac{\partial B_i(q^*)}{\partial q_{ij}} + \sigma_j^* = \sum_{k=1}^{n} \frac{\partial c_{ik}(q^*)}{\partial q_{ij}}, \text{ with } \sigma_j^* > 0, \tag{4.27}$$

and, therefore,

$$\sum_{k=1}^{n} \frac{\partial c_{ik}(q^*)}{\partial q_{ij}} > \sum_{k=1}^{n} \frac{\partial P_{ik}(q^*)}{\partial q_{ij}} + \omega_i \frac{\partial B_i(q^*)}{\partial q_{ij}}, \tag{4.28}$$

which means that the marginal costs are greater than the marginal utility associated with the financial donations plus altruism and this is a very bad situation.

Finally, if $\sum_{i=1}^{m} q_{ij}^* = \overline{d}_j$, then $\varepsilon_j^* > 0$, we have that:

$$\sum_{k=1}^{n} \frac{\partial P_{ik}(q^*)}{\partial q_{ij}} + \omega_i \frac{\partial B_i(q^*)}{\partial q_{ij}} = \sum_{k=1}^{n} \frac{\partial c_{ik}(q^*)}{\partial q_{ij}} + \varepsilon_j^*, \text{ with } \varepsilon_j^* > 0. \tag{4.29}$$

Therefore,

$$\sum_{k=1}^{n} \frac{\partial c_{ik}(q^*)}{\partial q_{ij}} < \sum_{k=1}^{n} \frac{\partial P_{ik}(q^*)}{\partial q_{ij}} + \omega_i \frac{\partial B_i(q^*)}{\partial q_{ij}}. \tag{4.30}$$

In this situation, the relevant marginal utility exceeds the marginal cost and this is a desirable situation.

Analogously, if we assume that the conservation of flow equation is active; that is, if $\sum_{j=1}^{n} q_{ij}^* = s_i$, then $\delta_i^* > 0$. As a consequence, we obtain:

$$\sum_{k=1}^{n} \frac{\partial P_{ik}(q^*)}{\partial q_{ij}} + \omega_i \frac{\partial B_i(q^*)}{\partial q_{ij}} = \sum_{k=1}^{n} \frac{\partial c_{ik}(q^*)}{\partial q_{ij}} + \delta_i^*, \text{ with } \delta_i^* > 0, \tag{4.31}$$

which means that, once again, the marginal utility associated with the financial donations plus altruism exceeds the marginal cost and this is the desirable situation.

From the above analysis of the Lagrange multipliers and marginal utilities at the equilibrium solution, we can conclude that the most convenient situation, in terms of the marginal utilities, is the one when $\sum_{i=i}^{m} q_{ij}^* = \overline{d}_j$ and $\sum_{j=1}^{n} q_{ij}^* = s_i$.

Taking into account the Lagrange multipliers, an equivalent variational formulation of problem (4.6) under constraints (4.4), (4.5), (4.7), and (4.8) is the following one:

$$\text{Find } (q^*, \delta^*, \sigma^*, \varepsilon^*) \in R_+^{mn+m+2n} :$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\sum_{k=1}^{n}\frac{\partial c_{ik}(q^*)}{\partial q_{ij}} - \sum_{k=1}^{n}\frac{\partial P_{ik}(q^*)}{\partial q_{ij}} - \omega_i \frac{\partial B_i(q^*)}{\partial q_{ij}} + \delta_i^* - \sigma_j^* + \varepsilon_j^*\right](q_{ij} - q_{ij}^*)$$
$$+\sum_{i=1}^{m}\left(s_i - \sum_{j=1}^{n} q_{ij}^*\right)(\delta_i - \delta_i^*)$$
$$+\sum_{j=1}^{n}\left(\sum_{i=1}^{m} q_{ij}^* - \underline{d}_j\right)(\sigma_j - \sigma_j^*) + \sum_{j=1}^{n}\left(\overline{d}_j - \sum_{i=1}^{m} q_{ij}^*\right)(\varepsilon_j - \varepsilon_j^*) \geq 0, \tag{4.32}$$

$$\forall q \in R_+^{mn},\ \forall \delta \in R_+^{m},\ \forall \sigma \in R_+^{n},\ \forall \varepsilon \in R_+^{n}.$$

4.4 The Algorithm and Case Study

Before we present the case study, we outline the algorithm that we utilize for the computations, notably, the Euler method of Dupuis and Nagurney (1993), since it nicely exploits the feasible set underlying variational inequality (4.32), which is simply the nonnegative orthant.

Recall that, as established in Dupuis and Nagurney (1993), for convergence of the general iterative scheme, which induces the Euler method, the sequence $\{a_\tau\}$ must satisfy: $\sum_{\tau=0}^{\infty} a_\tau = \infty$, $a_\tau > 0$, $a_\tau \to 0$, as $\tau \to \infty$. Conditions for convergence for a variety of network-based problems can be found in Nagurney and Zhang (1996) and Nagurney (2006).

Specifically, at iteration τ, the Euler method yields the following closed form expressions for the relief item flows and the Lagrange multipliers.

Explicit Formulae for the Euler Method Applied to the Game Theory Model

In particular, we have the following closed form expression for the relief item flows $i = 1, \ldots, m; j = 1, \ldots, n$, at each iteration:

$$q_{ij}^{\tau+1} = \max\{0, q_{ij}^{\tau} + a_\tau(\sum_{k=1}^{n} \frac{\partial P_{ik}(q^\tau)}{\partial q_{ij}} + \omega_i \frac{\partial B_i(q^\tau)}{\partial q_{ij}} - \sum_{k=1}^{n} \frac{\partial c_{ik}(q^\tau)}{\partial q_{ij}} - \delta_i^\tau + \sigma_j^\tau - \epsilon_j^\tau)\}; \tag{4.33}$$

the following closed form expressions for the Lagrange multipliers associated with the supply constraints (4.4), respectively, for $i = 1, \ldots, m$:

$$\delta_i^{\tau+1} = \max\{0, \delta_i^\tau + a_\tau(-s_i + \sum_{j=1}^{n} q_{ij}^\tau)\}; \tag{4.34}$$

the following closed form expressions for the Lagrange multipliers associated with the lower bound demand constraints (4.7), respectively, for $j = 1, \ldots, n$:

$$\sigma_j^{\tau+1} = \max\{0, \sigma_j^\tau + a_\tau(-\sum_{i=1}^{m} q_{ij}^\tau + \underline{d}_j)\}, \tag{4.35}$$

and the following closed form expressions for the Lagrange multipliers associated with the upper bound demand constraints (4.8), respectively, for $j = 1, \ldots, n$:

$$\epsilon_j^{\tau+1} = \max\{0, \epsilon_j^\tau + a_\tau(-\bar{d}_j + \sum_{i=1}^{m} q_{ij}^\tau)\}. \tag{4.36}$$

Our case study is inspired by a disaster consisting of a series of tornados that hit western Massachusetts on June 1, 2011 in the late afternoon. The largest tornado was measured at EF3. It was the worst tornado outbreak in the area in a century (see Flynn (2011)). A wide swath from western to central Massachusetts was impacted. According to the Western Massachusetts Regional Homeland Security Advisory Council report (2012): "The tornado caused extensive damage, killed 4 persons, injured more than 200 persons, damaged or destroyed 1,500 homes, left over 350 people homeless in Springfield's MassMutual Center arena, left 50,000 customers without power, and brought down thousands of trees." The same report notes that: FEMA estimated that 1,435 residences were impacted with the following breakdowns: 319 destroyed, 593 sustaining major damage, 273 sustaining minor damage, and 250 otherwise affected. FEMA estimated that the primary impact was damage to buildings and equipment with a cost estimate of $24,782,299. Total damage estimates from the storm exceeded $140 million, the majority from the destruction of homes and businesses.

Especially impacted were the city of Springfield and the towns of Monson and Brimfield. It has been estimated that in the aftermath, the Red Cross served about 11,800 meals and the Salvation Army about 20,000 meals (cf. Western Massachusetts Regional Homeland Security Advisory Council (2012)).

The network topology for our case study, Example 1, is depicted in Figure 2. The NGO nodes consist of the American Red Cross and the Salvation Army, respectively. The demand points correspond to Springfield, Monson, and Brimfield, respectively.

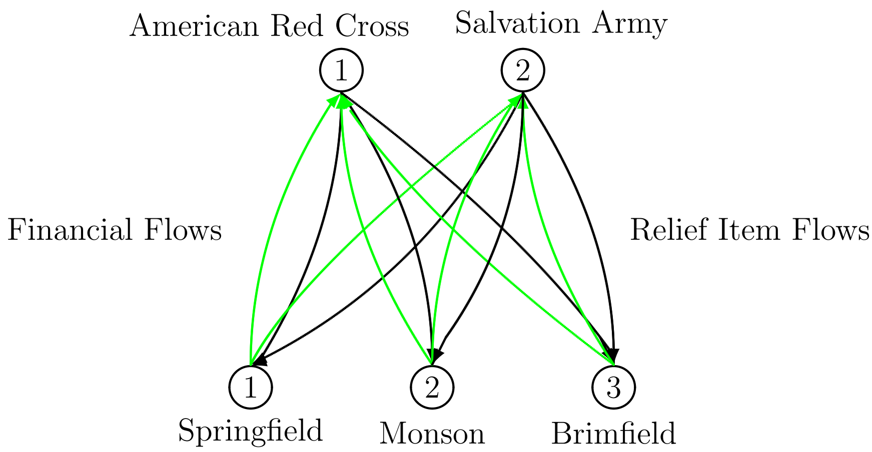

Figure 4.2: The Network Topology for the Case Study, Example 1

4.4.1 Example 1

The data for our case study, Example 1, are given below. The supplies of meals available for delivery to the victims are:

$$s_1 = 25,000, \quad s_2 = 25,000,$$

with the weights associated with the altruism benefit functions of the NGOs given by:

$$\omega_1 = 1, \quad \omega_2 = 1.$$

The financial funds functions are:

$$P_{11}(q) = 1000\sqrt{(3q_{11} + q_{21})}, \quad P_{12}(q) = 600\sqrt{(2q_{12} + q_{22})},$$

$$P_{13}(q) = 400\sqrt{(2q_{13} + q_{23})}, \quad P_{21}(q) = 800\sqrt{(4q_{21} + q_{11})},$$

$$P_{22}(q) = 400\sqrt{(2q_{22} + q_{12})}, \quad P_{23}(q) = 200\sqrt{(2q_{23} + q_{13})}.$$

The altruism functions are:

$$B_1(q) = 300q_{11} + 200q_{12} + 100q_{13}, \quad B_2(q) = 400q_{21} + 300q_{22} + 200q_{23}.$$

The cost functions, which capture distance from the main storage depots in Springfield, are:

$$c_{11}(q) = .15q_{11}^2 + 2q_{11}, \quad c_{12}(q) = .15q_{12}^2 + 5q_{12}, \quad c_{13}(q) = .15q_{13}^2 + 7q_{13},$$

$$c_{21}(q) = .1q_{21}^2 + 2q_{21}, \quad c_{22}(q) = .1q_{22}^2 + 5q_{22}, \quad c_{23}(q) = .1q_{23}^2 + 7q_{23}.$$

The demand lower and upper bounds at the three demand points are:

$$\underline{d}_1 = 10000, \quad \bar{d}_1 = 20000,$$

$$\underline{d}_2 = 1000, \quad \bar{d}_2 = 10000,$$

$$\underline{d}_3 = 1000, \quad \bar{d}_3 = 10000.$$

The Euler method was implemented in FORTRAN and a Linux system at the University of Massachusetts Amherst was used for the computations.

The algorithm was initialized as follows: all Lagrange multipliers were set to 0.00 and the initial relief item flows to a given demand point were set to the lower bound divided by the number of NGOs, which here is two.

The Euler method yielded the following Generalized Nash Equilibrium solution:

The equilibrium relief item flows are:

$$q^*_{11} = 3800.24, \quad q^*_{12} = 668.64, \quad q^*_{13} = 326.66,$$

$$q^*_{21} = 6199.59, \quad q^*_{22} = 1490.52, \quad q^*_{23} = 974.97.$$

Since none of the supplies are exhausted, the computed Lagrange multipliers associated with the supply constraints are:

$$\delta^*_1 = 0.00, \quad \delta^*_2 = 0.00.$$

Since the demand at the first demand point, which is the city of Springfield, is essentially at its lower bound, we have that:

$$\sigma^*_1 = 835.22,$$

with

$$\sigma^*_2 = 0.00, \quad \sigma^*_3 = 0.00.$$

All the Lagrange multipliers associated with the demand upper bound constraints are equal to zero, that is:

$$\epsilon^*_1 = \epsilon^*_2 = \epsilon^*_3 = 0.00.$$

In terms of the gain in financial donations, the NGOs receive the following amounts:

$$\sum_{j=1}^{3} P_{1j}(q^*) = 180,713.23, \quad \sum_{j=1}^{3} P_{2j}(q^*) = 168,996.78.$$

This is reasonable since the American Red Cross tends to have greater visibility post disasters than the Salvation Army through the media and that was the case post the Springfield tornadoes.

We then proceeded to solve the Nash equilibrium counterpart of the above Generalized Nash Equilibrium problem formulated as a variational equilibrium. The variational inequality for the Nash equilibrium is given in (4.13) and does not include the upper and lower bound demand constraints. We solved it using the Euler method but over the feasible set K as in (4.13).

The computed equilibrium relief item flows for the Nash equilibrium are:

$$q_{11}^* = 1040.22, \quad q_{12}^* = 668.64, \quad q_{13}^* = 326.66,$$

$$q_{21}^* = 2054.51, \quad q_{22}^* = 1490.52, \quad q_{23}^* = 974.97.$$

The Lagrange multipliers associated with the supply constraints are:

$$\delta_1^* = 0.00, \quad \delta_2^* = 0.00.$$

Observe that, without the imposition of the bounds on the demands, Springfield, which is demand point 1 and is a big city, receives only about one third of the volume of supplies (in this case, meals) as needed, and as determined by the Generalized Nash equilibrium solution.

The American Red Cross now garners financial donations of: 119,985.66, whereas the Salvation Army stands to receive financial donations equal to: 110,683.60. These values are significantly lower than the analogous ones for the Generalized Nash equilibrium model above. Hence, NGOs, by coordinating their deliveries of needed supplies, such as meals, can gain in terms of financial donations and attend to the victims' needs better by delivering in the amounts that have been estimated to be needed in terms of lower and upper bounds. This more general model, for which an optimization reformulation does not exist, in contrast to the model of Nagurney, Alvarez Flores, and Soylu (2016), nevertheless, supports the numerical result findings in the case study for Katrina therein.

4.4.2 Example 2

We now investigate the possible impact of the addition of a new disaster relief organization, such as a church-based one, or the Springfield Partners for Community Action, which also assisted in disaster relief, providing meals post the tornadoes. Hence, the network topology for case study, Example 2, is as in Figure 3. We refer to the added NGO as "Other." It is based in Springfield.

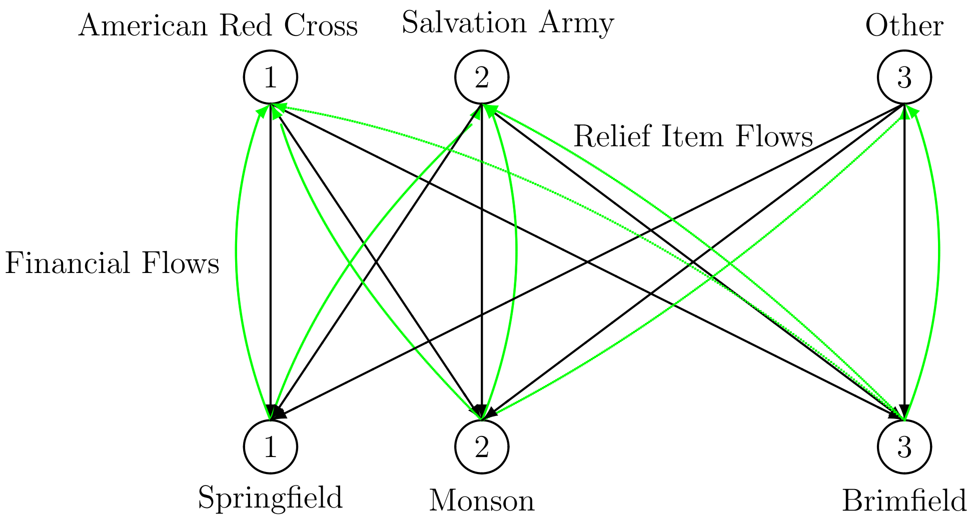

Figure 4.3: The Network Topology for the Case Study, Example 2

The data are as in Example 1 but with the original $P_{ij}(q)$ functions for the America Red Cross and the Salvation Army expanded as per below and the added data for the "Other" NGO also as given below.

The financial funds functions for are now:

$$P_{11}(q) = 1000\sqrt{(3q_{11} + q_{21} + q_{31})}, \quad P_{12}(q) = 600\sqrt{(2q_{12} + q_{22} + q_{32})},$$

$$P_{13}(q) = 400\sqrt{(2q_{13} + q_{23} + q_{33})}, \quad P_{21}(q) = 800\sqrt{(4q_{21} + q_{11} + q_{31})},$$

$$P_{22}(q) = 400\sqrt{(2q_{22} + q_{12} + q_{32})}, \quad P_{23}(q) = 200\sqrt{(2q_{23} + q_{13} + q_{33})},$$

with those for the new NGO:

$$P_{31}(q) = 400\sqrt{(2q_{31} + q_{11} + q_{21})}, \quad P_{32}(q) = 200\sqrt{(2q_{32} + q_{12} + q_{22})},$$

$$P_{33}(q) = 100\sqrt{(2q_{33} + q_{13} + q_{23})}.$$

The weight $\omega_3 = 1$ and the altruism/benefit function for the new NGO is:

$$B_3(q) = 200q_{31} + 100q_{32} + 100q_{33}.$$

The cost functions associated with the added NGO are:

$$c_{31}(q) = .1q_{31}^2 + q_{31}, \quad c_{32}(q) = .2q_{32}^2 + 5q_{32}, \quad c_{33}(q) = .2q_{33}^2 + 7q_{33}.$$

The Euler method converged to the following Generalized Nash Equilibrium solution:

The equilibrium relief item flows are:

$$q_{11}^* = 2506.97, \quad q_{12}^* = 667.85, \quad q_{13}^* = 325.59,$$

$$q_{21}^* = 4259.59, \quad q_{22}^* = 1489.98, \quad q_{23}^* = 974.45,$$

$$q_{31}^* = 3233.35, \quad q_{32}^* = 242.42, \quad q_{33}^* = 235.52.$$

Since none of the supplies are exhausted, the computed Lagrange multipliers associated with the supply constraints are:

$$\delta_1^* = 0.00, \quad \delta_2^* = 0.00, \quad \delta_3^* = 0.00.$$

The demand at the first demand point, which is the city of Springfield, is at the lower bound of 10000.00. Hence, we have that:

$$\sigma_1^* = 446.70,$$

with

$$\sigma_2^* = 0.00, \quad \sigma_3^* = 0.00.$$

All the Lagrange multipliers associated with the demand upper bound constraints are equal to zero, that is:

$$\epsilon_1^* = \epsilon_2^* = \epsilon_3^* = 0.00.$$

In terms of the gain in financial donations, the NGOs receive the following amounts:

$$\sum_{j=1}^{3} P_{1j}(q^*) = 173,021.70, \quad \sum_{j=1}^{3} P_{2j}(q^*) = 155,709.50, \quad \sum_{j=1}^{3} P_{3j}(q^*) = 60,504.14.$$

The volumes of relief items from the American Red Cross and the Salvation Army to Springfield are greatly reduced, as compared to the respective volumes in Example 1 and both original NGOs in Example 1 now experience a reduction in financial donations because of the increased competition for financial donations.

For completeness, we also solved the Nash equilibrium counterpart for Example 2.

The Nash equilibrium relief item flows are:

$$q^*_{11} = 1036.27, \quad q^*_{12} = 667.85, \quad q^*_{13} = 325.59,$$

$$q^*_{21} = 2051.17, \quad q^*_{22} = 1489.98, \quad q^*_{23} = 974.45,$$

$$q^*_{31} = 1009.61, \quad q^*_{32} = 242.42, \quad q^*_{33} = 235.52.$$

The financial donations of the NGOs are now the following:

$$\sum_{j=1}^{3} P_{1j}(q^*) = 129,037.42, \quad \sum_{j=1}^{3} P_{2j}(q^*) = 115,964.80, \quad \sum_{j=1}^{3} P_{3j}(q^*) = 43,07.16.$$

In Example 2 of our case study, we, again, see that the NGOs garner greater financial funds through the Generalized Nash Equilibrium solution, rather than the Nash equilibrium one. Moreover, the needs of the victims are met under the Generalized Nash Equilibrium solution.

4.5 Conclusions

In this chapter, we constructed a new Generalized Nash Equilibrium (GNE) model for disaster relief, which contains both logistical as well as financial funds aspects. The NGOs compete for financial funds through their visibility in the response to a disaster and provide needed supplies to the victims. A coordinating body imposes upper bounds and lower bounds for the supplies at the various demand points to guarantee that the victims receive the amounts at the points of demand that are needed, and without excesses that can add to the congestion and materiel convergence. The model is more general than the one proposed earlier by Nagurney, Alvarez Flores, and Soylu (2016) and no longer is it possible to reformulate the governing equilibrium conditions as an optimization problem.

Chapter 5

Conclusions

In this thesis we have focused our attention on two mathematical models.
In chapter 3 we proposed a mathematical model for the minimization of the total costs associated with organ transplants. In particular we presented the organ transplant network consisting of transplant centers and donor hospitals. We introduced the cost functions associated with transportations, with organ removals, with waste disposals, and with post-transplants. We determined the optimality conditions for the national health service and derived the variational inequality formulation. Then we studied the Lagrange theory related to the model in order to better understand the behavior of the transplant process, providing an interpretation of the Lagrange multipliers. Finally, we recalled the Euler method which has been applied to solve numerical examples.
In chapter 4 we presented a Generalized Nash Equilibrium model for post-disaster humanitarian relief. In particular we constructed the novel Generalized Nash Equilibrium model for disaster relief, which captures competition both on the financial funds side as well as on the logistics side and we identified the network structure. We presented the variational equilibrium framework and also proved the existence of an equilibrium solution. In addition, we provided the variational inequality formulation of a special case of the model, under the Nash equilibrium solution, in the absence of imposed

common demand constraints. Then we explored, through Lagrange analysis, the humanitarian organizations' marginal utilities when the equilibrium disaster relief flows are at the upper or the lower bounds of the imposed demands of the regulatory body or lie in between. Finally, we presented an algorithmic scheme and a case study, inspired by tornadoes that hit western Massachusetts in June 2011, with devastating impact.

In this thesis we have studied models without any time dependency, but in future works it is extremely important to take into account how the demands and the flows vary over time or to introduce uncertainty in the demands.

Bibliography

[1] O. Alagoz, A.J. Schaefer, M.S. Roberts, *Optimizing organ allocation and acceptance*, Handbook of Optimization in Medicine, 2013, 1-24.

[2] B. Balcik, B. Beamon, C. Krejci, K.M. Muramatsu, M. Ramirez, *Co-ordination in humanitarian relief chains: Practices, challenges and opportunities* International Journal of Production Economics, 120(1), 2010, 22-34.

[3] A. Barbagallo, P. Daniele, A. Maugeri, *Variational formulation for a general dynamic financial equilibrium problem: Balance law and liability formula*, Nonlinear Analysis, Theory, Methods and Applications, 75 (3), 2012, 1104-1123.

[4] J. Beliën, L. De Boeck, J. Colpaert, S. Devesse, F. Van den Bossche, *Optimizing the supply chain design for organ transplants*, Hub Research Paper, Economics & Management, 2011.

[5] A. Bensoussan, *Points de Nash dans le cas de fonctionelles quadratiques et jeux differentiels lineaires a N personnes* SIAM Journal on Control, 12, 1974, 460-499.

[6] A. Bentall, L.D. Cornell, J.M. Gloor, W.D. Park, W.D., M.J. Gandhi, J.L. Winters, M.F. Chedid, P.G. Dean, M.D. Stegall, *Five-year outcomes in living donor kidney transplants with a positive crossmatch*, American Journal of Transplantation, 13 (1), 2013, 76-85.

[7] R.D. Bloom, A.M. Doyle, Evaluation of the Kidney Transplant Candidate and Follow-Up of the Listed Patient. In: McKay D., Steinberg S. (eds) Kidney Transplantation: A Guide to the Care of Kidney Transplant Recipients. Springer, Boston, MA, 2010.

[8] M. E. Bruni. D. Conforti, N. Sicilia, S. Trotta, *A new organ transplantation location-allocation policy: a case study of Italy*, Health Care Manage Sci 9, 2006,125-142.

[9] V. Caruso, P. Daniele, *A network model for minimizing the total organ transplant costs*, European Journal of Operational Research, 2017, doi: 10.1016/j.ejor.2017.09.040

[10] V. Chankong, Y.Y. Haimes, *Multiobjective Decision Making: Theory and Methodology*, North-Holland, New York, 1983.

[11] J.B. Coles, J. Zhuang, *Decisions in disaster recovery operations: A game theoretic perspective on organization cooperation* Journal of Homeland Security and Emergency Management, 8(1), Article 35, 2011..

[12] C.W. Cox, *Manmade disasters: A historical review of terrorism and implications for the future*, Online Journal of Nursing, 13(1), January 8, 2008. Available at:
http://www.nursingworld.org/MainMenuCategories/ANAMarketplace
/ANAPeriodicals/OJIN/TableofContents/vol132008/No1Jan08/
ArticlePreviousTopic/ManmadeDisasters.html

[13] P. Daniele, *Variational inequalities for static equilibrium market*, in **Lagrangean Function and Duality, in Equilibrium Problems: Nonsmooth Optimization and Variational Inequality Models**. Kluwer Academic Publishers, F. Giannessi - A. Maugeri - P. Pardalos Eds., 2001, 43-58.

[14] P. Daniele, *Time-dependent spatial price equilibrium problem: Existence and stability results for the quantity formulation model*, Journal of Global Optimization, 28 (3-4), 2004, 283-295.

[15] P. Daniele, **Dynamic Networks and Evolutionary Variational Inequalities**, Edward Elgar Publishing, Cheltenham, UK, 2006.

[16] P. Daniele, S. Giuffrè, *Random Variational Inequalities and the Random Traffic Equilibrium Problem*, Journal of Optimization Theory and Applications, 167 (1), 2015, 363-381.

[17] P. Daniele, S. Giuffrè, G. Idone, A. Maugeri, *Infinite dimensional duality and applications*, Mathematische Annalen, 339 (1), 2007, 221-239.

[18] P. Daniele, A. Maugeri, A. Nagurney, *Cybersecurity Investments with Nonlinear Budget Constraints: Analysis of the Marginal Expected Utilities*, in Operations Research, Engineering, and Cyber Security, N.J. Daras, T.M. Rassias (eds.), Springer Optimization and Its Applications 113, 2017, 117-134.

[19] G.M. Danovitch, S. Hariharan, J.D. Pirsch, D. Rush, D. Roth, E. Ramos, R.C. Starling, C. Cangro, M.R. Weir; Clinical Practice Guidelines Committee of the American Society of Transplantation, *Management of the waiting list for cadaveric kidney transplants: report of a survey and recommendations by the Clinical Practice Guidelines Committee of the American Society of Transplantation.*, J Am Soc Nephrol., 2002.

[20] G. Debreu, *A social equilibrium existence theorem*, Proceedings of the National Academy of Sciences of the United States of America, 38, 1952, 886-893.

[21] P. Dupuis, A. Nagurney, *Dynamical systems and variational inequalities*, Annals of Operations Research 44, 1993, 9-42.

[22] F. Facchinei, C. Kanzow, *Generalized Nash equilibrium problems*, Annals of Operations Research, 175(1), 2010, 177-211.

[23] F. Facchinei, J.S. Pang, **Finite-dimensional variational inequalities and complementarity problems**, Berlin, Springer, 2003.

[24] A. Fischer, M. Herrich, K. Schonefeld, *Generalized Nash equilibrium problems - Recent advances and challenges*, Pesquisa Operacional, 34(3), 2014, 521-558.

[25] P.C. Fishburn, *Utility Theory for Decision Making*, John Wiley & Son, New York, 1970.

[26] D. Gabay, H. Moulin, *On the uniqueness and stability of Nash equilibria in noncooperative games*, In: Applied Stochastic Control of Econometrics and Management Science, A. Bensoussan, P. Kleindorfer, C.S. Tapiero, Editors, North-Holland, Amsterdam, The Netherlands, pp. 271-294, 1980.

[27] A. Gatignon, L. Van Wassenhove, A. Charles, *The Yogyakarta earthquake: Humanitarian relief through IFRC's decentralized supply chain*, International Journal of Production Economics, 126(1), 2010, 102-110.

[28] S. Giuffrè, A. Maugeri, D. Puglisi, *Lagrange multipliers in elastic-plastic torsion problem for nonlinear monotone operators*, Journal of Differential Equations, 259 (3), 2015, 817-837.

[29] J. Jahn, **Introduction to the Theory of Nonlinear Optimization**, Springer-Verlag, Berlin, 1996.

[30] R.L. Keeney, H. Raiffa, *Decisions with Multiple Objectives: Preferences and Value Tradeoffs*, Cambridge University Press, Cambridge, England, 1993.

[31] D. Kinderlehrer, G. Stampacchia, G., **An Introduction to Variational Inequalities and Their Applications**, Academic Press, New York, 1980.

[32] N. Koizumi, R. Ganesan, M. Gentili, C-H Chen, N. Waters, D. DasGupta, D. Nicholas, A. Patel, D. Srinivasan, K. Melancon, *Re-designing Organ Allocation Boundaries for Liver Transplantation in the United States*, Springer Proceedings in Mathematics & Statistics, vol 61., Springer, 2014.

[33] N. Kong, A.J. Schaefer, B. Hunsaker, M.S. Roberts, *Maximizing the efficiency of the US liver allocation system through region design*, Manag. Sci, 56 (12), 2010, 2111-2122.

[34] J.K. Kopinak, *Humanitarian assistance: Are effective and sustainability impossible dreams?*, Journal of Humanitarian Assistance, March 10, 2013.

[35] A.A. Kulkarni, U.V. Shanbhag, *On the variational equilibrium as a refinement of the generalized Nash equilibrium*, Automatica, 48, 2012, 45-55.

[36] J.L. Lions, G. Stampacchia, *Le disuguaglianze variazionali*, Comunicazioni su Matematica Pura e Applicata , 20, pp 493-519, 1967.

[37] J.P. Luna, *Decomposition and Approximation Methods for Variational Inequalities, with Applications to Deterministic and Stochastic Energy Markets*, PhD thesis, Instituto Nacional de Matematica Pura e Aplicada, Rio de Janeiro, Brazil2013.

[38] A. Malani, T. Philipson, G. David, *Theories of firm behavior in the non-profit sector. A synthesis and empirical evaluation*, In: The Governance of Not-for-Profit Organizations. E.L. Glaeser, Editor, University of Chicago Press, Chicago, Illinois, pp. 181-215, 2003.

[39] D.P. Moynihan, *The response to Hurricane Katrina*, International Risk Governance Council, Geneva, Switzerland, 2009.

[40] L. Muggy, *Quantifying and Mitigating Decentralized Decision Making in Humanitarian Logistics Systems*, PhD Dissertation, Kansas State University, Manhattan, Kansas, 2015.

[41] L. Muggy, J.L. Heier Stamm, *Game theory applications in humanitarian operations: A review*, Journal of Humanitarian Logistics and Supply Chain Management, 4(1), 2014, 4-23.

[42] A. Nagurney, **Network Economics: A Variational Inequality Approach**, second and revised edition, Kluwer Academic Publishers, Boston, Massachusetts, 1999.

[43] A. Nagurney, *Supply Chain Network Economics: Dynamics of Prices, Flows, and Profits*, Edward Elgar Publishing. Cheltenham, England, 2006.

[44] A. Nagurney, *Freight service provision for disaster relief: A competitive network model with computations*, In: Dynamics of Disasters: I.S. Kotsireas, A. Nagurney, and P.M. Pardalos, Editors, Springer International Publishing Switzerland, pp. 207-229, 2016.

[45] A. Nagurney, E. A. Flores, P. Daniele, V. Caruso, *A Variational Equilibrium Network Framework for Humanitarian Organizations in Disaster Relief: Effective Product Delivery Under Competition for Financial Funds*, submitted.

[46] A. Nagurney, W. Alvarez Flores, C. Soylu, *A Generalized Nash Equilibrium model for post-disaster humanitarian relief* Transportation Research E, 95, 2016, 1-18.

[47] A. Nagurney, J. Dong, *Supernetworks: Decision-Making for the Information Age* Edward Elgar Publishing, Cheltenham, England, 2002.

[48] A. Nagurney, Q. Qiang, *Fragile Networks: Identifying Vulnerabilities and Synergies in an Uncertain World*, John Wiley & Sons, Hoboken, New Jersey, 2009.

[49] A. Nagurney, S. Shukla, *Multifirm Models of Cybersecurity Investment Competition vs. Cooperation and Network Vulnerability*, European Journal of Operational Research 269(2), 2017, 588-600.

[50] A. Nagurney, M. Yu, D. Besik, *Supply chain network capacity competition with outsourcing: A variational equilibrium framework*, In press in Journal of Global Optimization, 2017.

[51] A. Nagurney, D. Zhang, *Projected Dynamical Systems and Variational Inequalities with Applications*, Kluwer Academic Publishers, Boston, Massachusetts, 1996.

[52] J.F. Nash, *Equilibrium points in n-person games*, Proceedings of the National Academy of Sciences, USA, 36, 1950, 48-49.

[53] J.F. Nash, *Noncooperative games*, Annals of Mathematics, 54, 1951, 286-298.

[54] A.S. Natsios, *NGOs and the UN system in complex humanitarian emergencies: conflict or cooperation?*, Third World Quarterly, 16(3), 1995, 405-419.

[55] J. Ratcliffe, T. Young, M. Buxton, T. Eldabi, R. Paul, A. Burroughs, G. Papatheodoridis, K. Rolles, *A Simulation Modelling Approach to Evaluating Alternative Policies for the Management of the Waiting List for Liver Transplantation*, Health Care Management Science, 2001.

[56] Rogrilie, Matthew, Peng Shi Amy Wen, *Optimizing the Effectiveness of Organ Allocation*, Academic Journal, Vol. 28 Issue 2, p117, 2007.

[57] L. Roopanarine, *How pre-positioning can make emergency relief more effective*, The Guardian, January 17, 2013.

[58] S. Rose-Ackerman, *Charitable giving and "excessive" fundraising*, The Quarterly Journal of Economics, 97(2), 1982, 193-212.

[59] A. Signorini, *Questioni di elasticità non linearizzata e semilinearizzata*, Rendiconti di Matematica e delle sue applicazioni, 5, vol. 18, pp. 95-139, 1959.

[60] G. Stampacchia, *Formes bilineaires coercitives sur les ensembles convessi*, Comptes rendus hebdomadaires des sedute de l'Académie des Sciences, 258, pp 4413-4416, 1964.

[61] R.W. Strong, S.V. Lynch, T.H. Ong, H. Matsunami, Y. Koido, G.A. Balderson, *Successful liver transplantation from a living donor to her son*, New England Journal of Medicine, 322 (21), 1990, 1505-1507.

[62] D. Sumpf, V. Isaila, K. Najjar, *The impact of the Syria crisis on Lebanon*, In: Dynamics of Disasters: Key Concepts, Models, Algorithms, and Insights, I.S. Kotsireas, A. Nagurney, and P.M. Pardalos, Editors, Springer International Publishing Switzerland, pp. 269-308, 2016.

[63] F. Toyasaki, P. Daniele, T. Wakolbinger, *A variational inequality formulation of equilibrium models for end-of-life products with nonlinear constraints*, European Journal of Operational Research 236 (1), 2014, 340-350.

[64] F. Toyasaki, T. Wakolbinger, *Impacts of earmarked private donations for disaster fundraising*, Annals of Operations Research, 221, 2014, 427-447.

[65] L.N. Van Wassenhove, *Blackett memorial lecture. Humanitarian aid logistics: Supply chain management in high gear*, Journal of the Operational Research Society, 57(5), 2006, 475-489.

[66] A. Von Heusinger, *Numerical Methods for the Solution of the Generalized Nash Equilibrium Problem*, PhD Dissertation, University of Wurtburg, Germany, 2009.

[67] C. Watson, A. Caravani, T. Mitchell, J. Kellett, K. Peters, *10 things to know about finance for reducing disaster risk*, Overseas Development Institute, London, England, 2015.

[68] P.L. Yu, *Multiple Criteria Decision Making: Concepts, Techniques, and Extensions*, Plenum Press, New York, 1985.

[69] Ö. Yilmaz, *Kidney exchange: Further utilization of donors via listed exchange*, Journal of Mathematical Economics, 51, 2014, 178-186.

[70] J. Zhuang, G. Saxton, H. Wu, *Publicity vs. impact in nonprofit disclosures and donor preferences: A sequential game with one nonprofit organization and N donors*, Annals of Operations Research, 221(1), 2014, 469-491.

[71] http://mat.gsia.cmu.edu/classes/QUANT/NOTES/chap4.pdf