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New Advances on Multiple Criteria Hierarchy Process and the Choquet integral preference model

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List of the publications of Salvatore Corrente

- S.Corrente, M. Doumpos, S. Greco, R. Słowiński, C. Zopounidis. Multiple Criteria Hierarchy Process for Sorting Problems Based on Ordinal Regression with Additive Value Functions. Annals of Operations Research. In Press. DOI:10.1007/s10479-015-1898-1.

- S. Angilella, S. Corrente, S. Greco, R. Słowiński. Robust Ordinal Regression and Stochastic Multiobjective Acceptability Analysis in Multiple Criteria Hierarchy Process for the Choquet integral preference model. Omega. In Press. DOI:10.1016/j.omega.2015.10.010.

- S. Angilella, S. Corrente, S. Greco (2015). Stochastic Multiobjective Acceptability Analysis for the Choquet integral preference model and the scale construction problem. European Journal of Operational Research, 240 (1), 172-182.

- S. Corrente, S. Greco, A. Ishizaka. Combining Analytical Hierarchy Process and Choquet integral within Non Additive Robust Ordinal Regression. Omega. In Press. DOI:10.1016/j.omega.2015-07-003.

- J. Branke, S. Corrente, S. Greco, R. Słowiński, P. Zielniewicz. Using Choquet Integral as Preference Model in Interactive Evolutionary Multiobjective Optimization. European Journal of Operational Research. In Press. DOI: 10.1016/j.ejor.2015.10.027.

Introduction

What we mean by the terms Multiple Criteria Decision Analysis and Multiple Criteria Decision Aiding[1]? In the following, without seek of completeness, we report some statements, given by researchers well-known in the field, defining what Multiple Criteria Decision Analysis and Multiple Criteria Decision Aiding are, and which are their main objectives.

- Bell, 1977 [12]:

  Almost all the issues that decision makers face in actuality involve multiple objectives that conflict in some measure with each other. In such issues, decisions that serve some objectives well will generally satisfy other objectives less well than alternative decisions, which, however, would not be so satisfactory for the first group. The decision maker then must select from among the possible decisions the one that somehow establishes the best mix of outcomes for his multiple conflicting objectives. ... Such problems include the use of energy resources, the management of the environment, the development of water resources, and the expansion of regional development.

- Keeney and Raiffa, 1993 [109]:

  The theory of decision analysis is designed to help the individual make a choice among a set of prespecified alternatives ... The aim of the analysis is to get your head straightened out.

- Roy, 2005 [142]:

  Decision aiding is the activity of the person who, through the use of explicit but not necessarily completely formalized models, helps obtain elements of responses to the questions posed by a stakeholder in a decision process.

[1]Multiple Criteria Decision Analysis and Multiple Criteria Decision Aiding are both abbreviated by "MCDA". In the following, we shall use the acronym MCDA referring to both Multiple Criteria Decision Analysis and Multiple Criteria Decision Aiding indifferently.

- Belton and Stewart, 2000 [14]:

> *... we use the expression MCDA as an umbrella term to describe a collection of formal approaches which seek to take explicit account of multiple criteria in helping individuals or groups explore decisions that matter.*

- Saaty, 2005 [148]:

> *The purpose of decision-making is to help people make decisions according to their own understanding. ... decision-making is the most frequent activity of all people all the time...*

In MCDA (see [12, 14, 51, 95] for some books providing surveys on MCDA), a set of alternatives $A = \{a, b, c, \ldots\}$ is evaluated on the basis of a finite and consistent [141] family of criteria $G = \{g_1, \ldots, g_n\}$ in order to deal with a choice, ranking or sorting problem. Choice problems consist into selecting one or more alternatives from A considered the best among the considered ones; ranking problems consist into rank ordering all the alternatives from the best to the worst while sorting problems consist into assigning each alternative from A to one or more contiguous and preferentially ordered classes or categories. Each criterion can have an increasing or a decreasing direction of preference. If a criterion g_j, $j = 1, \ldots, n$, has an increasing direction of preference, then the higher the evaluation of a on g_j ($g_j(a)$), the better a is with respect to criterion g_j while, if g_j has a decreasing direction of preference then, the higher $g_j(a)$ the worse a is with respect to criterion g_j. In the following, for the sake of simplicity and without loss of generality, we suppose that all criteria have an increasing direction of preference.

When looking at the evaluations of the alternatives on all criteria simultaneously, the only objective information that can be obtained is the dominance relation for which, an alternative a dominates an alternative b if a is at least as good as b on all criteria ($g_j(a) \geq g_j(b)$ for all $j = 1, \ldots, n$) and strictly better for at least one criterion (there exists at least one $j \in \{1, \ldots, n\}$, such that $g_j(a) > g_j(b)$). In general, the dominance relation is really poor since, very often, comparing two alternatives a and b, a is better than b on some criteria, while b is better than a on the other criteria. For this reason, one needs to aggregate the evaluations got by the alternatives on the considered criteria by means of some aggregation method. Three different families of aggregation methods are known in MCDA that are, the Multiple Attribute Value Theory (MAVT) [109], the outranking methods (in particular ELECTRE [55] and PROMETHEE [27]) and the Dominance Based Rough Set Approach (DRSA) [82]. MAVT assigns to each alternative a a real number $U(a)$ being representative of the goodness of

a with respect to the problem at hand; outranking methods are based on a pairwise binary relation S where aSb means that a is at least good as b; DRSA aggregates the preferences of the Decision Maker (DM) through the use of a set of "if,...,then" decision rules expressed in a natural and easily understandable language for the DM. All these methods are based on several parameters and these could be obtained in a direct or an indirect way. The DM provides a direct preference information if he gives directly values to all parameters involved in the model, while the DM provides an indirect preference information if he gives some preference information such as comparisons between alternatives (for example, a is preferred to b or a is indifferent to b) or comparison between criteria with respect to their importance (for example, g_i is more important than g_j or g_i is as important as g_j), from which parameters compatible with these preferences can be inferred. Since the direct preference information asks a great cognitive effort to the DM, it is advisable, in general, to adopt the indirect preference information. Considering the indirect way of providing preference information, more than one set of parameters could be compatible with these preferences and, for such a reason, choosing only one of these sets of parameters could be considered arbitrary or even meaningless. In order to provide more robust conclusions with respect to the problem at hand taking into account the plurality of sets of parameters compatible with the preference information provided by the DM, the Robust Ordinal Regression (ROR) (see [86] for the seminal paper on ROR and [35, 36] for two recent surveys on ROR) and the Stochastic Multiobjective Acceptability Analysis (SMAA) (see [113] for the paper introducing SMAA and [164] for a survey on SMAA) are used in practice. Both of them explore the whole set of parameters compatible with the preferences provided by the DM even if in different ways. ROR builds two binary preference relations, one necessary and one possible. The necessary preference relation holds between two alternatives a and b if a is at least as good as b for all sets of parameters compatible with the preferences of the DM, while the possible preference relation holds between a and b if a is at least as good as b for at least one set of parameters compatible with the DM's preferences. SMAA explores, instead, the whole set of parameters compatible with the preferences of the DM, computing for each alternative a the frequency of attaining a certain position in the final ranking or the frequency with which an alternative a is preferred to another alternative b.

MCDA methods have been applied to deal with financial problems [47, 159], natural resource management problems [120], energy planning problems [44, 175], environmental problems [110, 117, 118] etc.

In this thesis we have dealt with two MCDA issues, that are, the hierarchy of criteria and the interaction between criteria.

In all real world decision making problems, the evaluation criteria are not sited all at the same level but they are structured in a hierarchical way. This means that it is possible to highlight a root criterion, some criteria at the second level descending from the root criterion, other criteria descending from the criteria at the second level and so on. In the same real world problems, the evaluation criteria are not mutually preferentially independent but they can present a certain degree of positive or negative interaction. In particular, two criteria g_i and g_j are positively interacting if the importance of the set composed of these two criteria is greater than the sum of the importance assigned to the two criteria singularly. Analogously, g_i and g_j are negatively interacting if the importance assigned to the set composed of these two criteria is smaller than the sum of the importance assigned to the pair of criteria taken singularly. For example, in evaluating a sport car and taking into account criteria such as acceleration, maximum speed and price, on one hand, acceleration and maximum speed can be considered as negatively interacting criteria while, on the other hand, maximum speed and price can be considered as positively interacting criteria. Indeed, a car presenting a high good acceleration has, in general, a good maximum speed and, for this reason, the importance assigned to the pair of criteria should be lower than the sum of the importance assigned to the two criteria considered singularly in order to avoid to overestimate a car having a good acceleration and a good maximum speed. At the same time, a car having a good maximum speed and a low price is well appreciated by the DM since, in general, a car having a good maximum speed has also a very high price. In this case, the importance assigned to this pair of criteria should be greater than the sum of the importance assigned to the criteria taken singularly.

Regarding the hierarchy of criteria, on the basis of the recently introduced Multiple Criteria Hierarchy Process (MCHP) framework [37], we have extended the well known sorting method UTADIS [43] to deal with sorting problems in which criteria are structured in a hierarchical way. Moreover, the ROR has been also applied to take into account the plurality of sets of parameters compatible with the indirect preference information provided by the DM.

Related to the hierarchy of criteria, we have also proposed a method putting together the Choquet integral preference model, the MCHP, the ROR and the SMAA. On one hand, the capabilities of taking into account interactions between criteria of the Choquet integral preference model have been used to deal with problems presenting a hierarchical structure of interacting criteria. On the other hand, on the basis of the indirect preference information, we applied the ROR and the SMAA to get robust recommendations with respect to the considered problem taking into account all the sets of parameters compatible with the indirect preference information provided by the DM.

Regarding the interaction between criteria, and in particular the Choquet integral preference model, three contributions, that we briefly summarize in the following, have been proposed:

- Since, as we shall explain in detail in Section 1.2, the application of the Choquet integral asks that the evaluations of the alternatives on the considered criteria are expressed on the same scale, in the first contribution we proposed a way to build this common scale. Because more than one common scale could be built by the proposed methodology, we take into account the plurality of these common scales by using SMAA;
- In the second contribution, the Analytic Hierarchy Process (AHP) (see [146] and Section 1.3) is applied to build the common scale in which the Choquet integral is based. Since the application of the AHP asks plenty of comparisons between alternatives on each considered criterion, we proposed a parsimonious way of asking preference information to the DM with respect to this point;
- In the third contribution, an interactive evolutionary multiobjective optimization method based on the Choquet integral preference model, and called NEMO-II-Ch, has been proposed. Based on the well known evolutionary multiobjective method NSGA-II [39], NEMO-II-Ch uses the Choquet integral preference model and the ROR to address the search in the most interesting region of the Pareto front for the DM.

The thesis is organized as follows: in chapter 1, we present some basic MCDA concepts and, in particular, MAVT (section 1.1), the Choquet integral preference model (section 1.2), the AHP (section 1.3), the ROR (section 1.4) and the SMAA methodology (section 1.5); the chapter 2 contains the two contributions related to the hierarchy of criteria that are "Multiple Criteria Hierarchy Process for Sorting Problems Based on Ordinal Regression with Additive Value Functions" (section 2.1) and "Robust Ordinal Regression and Stochastic Multiobjective Acceptability Analysis in Multiple Criteria Hierarchy Process for the Choquet integral preference model" (section 2.2); in chapter 3 we present in detail the three contributions related to the interaction issue, that are, "Stochastic Multiobjective Acceptability Analysis for the Choquet integral preference model and the scale construction problem" (section 3.1), "Combining Analytical Hierarchy Process and Choquet integral within Non Additive Robust Ordinal Regression" (section 3.2) and "Using Choquet Integral as Preference Model in Interactive Evolutionary Multiobjective Optimization" (section 3.3). Final considerations contained in chapter 4 conclude the thesis.

Contents

Chapter 1

Basic MCDA concepts

1.1 Multiple Attribute Value Theory

As already mentioned in the introduction, one of the most used approach to aggregate the performances of an alternative on the considered criteria is that one of assigning a real number being representative of the goodness of the alternative with respect to the problem at hand. Preference theory, and in this case Multiple Attribute Value Theory [109], studies how to construct appropriate functions for decision making representing the preferences of the DM. It is possible to distinguish between preferences under conditions of certainty or risk and over alternatives described by a single attribute or by multiple attributes. We shall refer to a preference representation function under certainty as a value function, and to a preference representation function under risk as a utility function [109]. In the following, we shall consider multiple attribute value functions.

Denoting by $\succsim$ the DM's preference relation over $\mathcal{I}$, where $\mathcal{I} = \prod_{j=1}^{n} \mathcal{I}_j$, $\mathcal{I}_j$ is the set of the performances got by the alternatives from A on criterion g_j and $a \succsim b$ means that a is at least as good as b, in the following we provide conditions ensuring the existence of a function $U : \mathcal{I} \to \mathbb{R}$ such that $a \succsim b$ if and only if $U(\overline{g}(a)) \geq U(\overline{g}(b))$, with $\overline{g}(a), \overline{g}(b) \in \mathcal{I}$ and $\overline{g}(a) = (g_1(a), \ldots, g_n(a))$, for all $a \in A$. A necessary condition for the existence of such a function is that $\succsim$ is a weak order (complete and transitive binary relation). If A is uncountable, a second condition (and then both are necessary and sufficient) is that $A/\sim$ contains a countable order-dense subset where $\sim$ is the symmetric part of $\succsim$. This condition is known as the Birkhoff-Milgram theorem. The theorem, as well as its proof, can be found in [138]. The interested reader is also referred to [42] and [84]).

The most common form for the value function U, and the most used in real world applications, is the additive one. In an additive representation, a real value is assigned to each alternative a by:

$$U(a) = \sum_{j=1}^{n} u_j(g_j(a))$$

where u_j are single attribute non-decreasing value functions over $\mathcal{I}_j$. Defining $x_j^* = \max_{a \in A} g_j(a)$ and $x_{j,*} = \min_{a \in A} g_j(a)$ the best and the worst evaluations an alternative belonging to A can get on criterion g_j, the value function U is normalized in the interval $[0,1]$ by imposing that $\sum_{j=1}^{n} u_j(x_j^*) = 1$ and $u_j(x_{j,*}) = 0$ for all $j = 1, \ldots, n$.

The condition ensuring the existence of an additive value function is the *mutual preference independence* of the set of criteria G [109, 174]. We say that the set of criteria $T \subseteq G$ is *preferentially independent* of $G \setminus T$ if, for all $a_T, b_T \in \prod_{j \in T} \mathcal{I}_j$, and for all $c_{G \setminus T}, d_{G \setminus T} \in \prod_{j \in G \setminus T} \mathcal{I}_j$,

$$(a_T, c_{G \setminus T}) \succsim (b_T, c_{G \setminus T}) \Leftrightarrow (a_T, d_{G \setminus T}) \succsim (b_T, d_{G \setminus T})$$

that is, the preference of $(a_T, c_{G \setminus T})$ over $(b_T, c_{G \setminus T})$ does not depend on $c_{G \setminus T}$. The whole set of criteria G is said to be *mutually preferentially independent* if T is preferentially independent of $G \setminus T$ for every $T \subseteq G$.

While the additive value function is an attractive choice for practical applications of multiattribute decision making, the assessment of the single attribute value functions relies on techniques that are cumbersome in practice, and that force the DM to make explicit tradeoffs between two or more criteria. Two assessment procedures for ordinal additive value functions are illustrated in [109]. Using a value function U, one gets a complete order among the considered alternatives, differently from the outranking methods in which in addition to the preference (a is preferred to b if aSb and $not(bSa)$) and indifference relations (a is indifferent to b if aSb and bSa) also an incomparability relation (a is incomparable with b if $not(aSb)$ and $not(bSa)$) is defined.

1.2 The Choquet integral preference model

The mutual preference independence between criteria introduced in the previous section, is not always fulfilled in practice. As a consequence, in this cases, an additive value function is not able to represent the preferences provided by the DM.

Inspired by [77], let us suppose that four trainees learning to drive military vehicles are evaluated on three criteria: Precision (P), Rapidity (R) and Communication (C). The performances of the trainees on the three criteria are expressed on a $[0,1]$ scale and they are shown in Table 1.1. The instructor thinks that both P and R are really important criteria but there exists a certain compensation between them. Comparing $Williams$ and $Johnson$, the instructor observes that both trainees have

Table 1.1: Trainees' evaluations

Trainee	Precision (P)	Rapidity (R)	Communication (C)
Johnson	0.7	0.9	0.2
Williams	0.7	0.7	0.6
Brown	0.2	0.9	0.2
Davis	0.2	0.7	0.6

good performances on P and R but $Williams$ has a better performance than $Johnson$ in C, so he prefers $Williams$ over $Johnson$. Comparing $Brown$ and $Davis$, the instructor observes that their performances on P are very low, so, because he thinks that P and R are more important than C, he states that $Brown$ is preferred to $Davis$ for his better performance on R. It is easy to observe that the set of criteria {R, C} is not preferentially independent from criterion P and, therefore, the whole set of criteria {P,R,C} is not mutually preferentially independent. Consequently, the preferences of the instructor can not be represented by an additive value function. Indeed, denoting by u_P, u_R and u_C the marginal value functions on P, R and C,

- the preference $Williams \succ Johnson$, is translated to the constraint

$$u_P(0.7) + u_R(0.7) + u_C(0.6) > u_P(0.7) + u_R(0.9) + u_C(0.2), \tag{1.1}$$

- while the preference $Brown \succ Davis$ is translated to the constraint

$$u_P(0.2) + u_R(0.9) + u_C(0.6) > u_P(0.2) + u_R(0.7) + u_C(0.6). \tag{1.2}$$

The two inequalities lead to the following contradiction:

$$u_R(0.7) - u_R(0.9) + u_C(0.6) - u_C(0.2) > 0 > u_R(0.7) - u_R(0.9) + u_C(0.6) - u_C(0.2).$$

In this case, the compensation between P and R observed by the instructor can be paraphrased saying that the two criteria are negatively interacting, that is, the importance assigned to this pair of criteria should be lower than the sum of the importances assigned to the two criteria considered alone.

In order to take into account the possible interactions (positive or negative) between criteria, in MCDA non-additive integrals are used and, in particular, the Choquet integral [31] and the Sugeno integral [162] as well as their generalizations, that are the bipolar Choquet integral [71, 72], the bipolar Sugeno integral [89] and the level dependent Choquet integral [80]. Other methods dealing with the possible interactions between criteria are the multilinear value functions [109] and the UTA^{GMS}-INT method [88]. Let us mention that the interaction between criteria has been also considered in outranking methods and, in particular, in ELECTRE methods [52] and in the PROMETHEE methods [34]. In the following, we shall describe the Choquet integral preference model since four out of the five contributions introduced in the thesis are based on it.

Differently from the weighted sum in which a weight w_j is assigned to each criterion g_j, the Choquet integral preference model is based on a capacity (or non-additive measure) $\mu : 2^G \rightarrow [0,1]$ that assigns a weight to each subset of G and that has to satisfy the following normalization and monotonicity constraints:

1a) $\mu(\emptyset) = 0$, $\mu(G) = 1$,

2a) $\mu(A) \leq \mu(B)$, for all $A \subseteq B \subseteq G$.

A capacity is additive if $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A \cap B = \emptyset$. If a capacity is additive, then it is sufficient defining the m coefficients $\mu(\{g_1\}), \ldots, \mu(\{g_m\})$.

Given a capacity μ, and $x = (x_1, \ldots, x_n) \in \mathcal{I}$ such that $x_j = g_j(x) \geq 0, \forall j = 1\ldots, n$, the Choquet integral of x with respect to μ is defined by

$$C_\mu(x) = \int_0^{+\infty} \mu(\{j \in G : x_j \geq t\})dt, \tag{1.3}$$

or, equivalently, as

$$C_\mu(x) = \sum_{j=1}^{n} x_{(j)} \left[\mu(A_{(j)}) - \mu(A_{(j+1)})\right] = \sum_{j=1}^{n} \left[x_{(j)} - x_{(j-1)}\right] \mu(A_{(j)}), \tag{1.4}$$

where $0 = x_{(0)} \leq x_{(1)} \leq \ldots \leq x_{(n)}$, $A_{(j)} = \left\{i \in G : x_i \geq x_{(j)}\right\}$ and $A_{(n+1)} = \emptyset$.
Going back to the example above, and using the definition of the Choquet integral in (1.4), the preferences of the instructor can be translated to the following two inequalities:

$$Williams \succ Johnson \ \Leftrightarrow \ 0.2\mu(\{R\}) + 0.4\mu\left(\{P,R\}\right) < 0.4, \tag{1.5}$$

and

$$Brown \succ Davis \ \Leftrightarrow \ 0.2\mu\left(\{R,C\}\right) < 0.3\mu\left(\{R\}\right). \tag{1.6}$$

Inequalities (1.5) and (1.6) are not in contradiction since a capacity such that $\mu(\{R\}) = \mu(\{P\}) = 0.5$, $\mu(\{C\}) = 0.1$, $\mu(\{R,P\}) = 0.7$, $\mu(\{R,C\}) = \mu(\{P,C\}) = 0.7$ and $\mu(\{R,P,C\}) = 1$ is compatible with the two inequalities and, consequently, the Choquet integral is able to describe the preferences of the instructor.

Considering the Möbius representation $a : 2^G \to \mathbb{R}$ of the capacity μ [140], such that

$$\mu(T) = \sum_{R\subseteq T} a(R), \ \text{ for all } T \subseteq G$$

constraints $1a$) and $2a$) can be replaced by the constraints

1b) $a(\emptyset) = 0, \ \sum_{T\subseteq G} a(T) = 1,$

2b) $\forall\, j \in G$ and $\forall S \subseteq G \setminus \{j\}, \ \sum_{T\subseteq S} a(T \cup \{j\}) \geq 0,$

and the Choquet integral can be expressed in a linear form as follows [66]:

$$C_\mu(x) = \sum_{T\subseteq G} a(T) \min_{i\in T} g_i\left(x\right).$$

The application of the Choquet integral requests the knowledge of a great number of parameters. Indeed, one needs $2^n - 2$ parameters (the value assigned by μ to each subset of G with the exception of $\emptyset$ and the same set G because $\mu(\emptyset) = 0$, and $\mu(G) = 1$). In order to reduce this number of parameters, in general k-additive capacities are considered [69]; a capacity is said *k-additive* if $a(T) = 0$ with $T \subseteq G$, when $|T| > k$[1]. In many real world decision making problems, it is sufficient to consider 2-additive capacities and, in this case, positive and negative interactions are considered only between couples of criteria neglecting any interaction among triples, quadruplets and generally m-tuples, (with $m > 2$) of criteria. In this way, the DM has to provide $n + \binom{n}{2}$ parameters (in terms of Möbius representation, a value $a(\{i\})$ for every criterion i and a value $a(\{i,j\})$ for every couple of distinct

[1]Observe that a 1-additive capacity is the common additive capacity

criteria $\{i,j\}$). With respect to a 2-additive capacity, the inverse transformation to obtain the fuzzy measure $\mu(R)$ from the Möbius representation is defined as:

$$\mu(R) = \sum_{i \in R} a(\{i\}) + \sum_{\{i,j\} \subseteq R} a(\{i,j\}), \ \forall R \subseteq G. \tag{1.7}$$

With regard to 2-additive measures, properties **1b)** and **2b)** have the following formulations

1b) $a(\emptyset) = 0, \ \sum_{i \in G} a(\{i\}) + \sum_{\{i,j\} \subseteq G} a(\{i,j\}) = 1,$

2b) $\begin{cases} a(\{i\}) \geq 0, \ \forall i \in G, \\ a(\{i\}) + \sum_{j \in T} a(\{i,j\}) \geq 0, \ \forall i \in G \text{ and } \forall \, T \subseteq G \setminus \{i\}, \, T \neq \emptyset. \end{cases}$

while the Choquet integral of $x \in A$ is given by:

$$C_\mu(x) \;=\; \sum_{i \in G} a(\{i\})\,(g_i(x)) + \sum_{\{i,j\} \subseteq G} a(\{i,j\}) \min\{g_i(x), g_j(x)\}. \tag{1.8}$$

In case of non-additive measures, the importance of a criterion does not depend on itself only but also on its contribution to all coalitions of criteria in G. For this reason, the Shapley value [154] and the Murofushi index [128] have been introduced. The Shaplex value gives the importance of criterion g_i, $i = 1, \ldots, n$, and it is given by

$$\varphi(i) = \sum_{T \subseteq G : i \notin T} \frac{(|G-T|-1)!|T|!}{|G|!} [\mu(T \cup \{i\}) - \mu(T)], \tag{1.9}$$

while the interaction index for a couple of criteria $\{i,j\} \subseteq G$, expresses the importance assigned to the pair of criteria $\{g_i, g_j\}$ and is computed as follows:

$$\varphi(\{i,j\}) = \sum_{T \subseteq G : i,j \notin T} \frac{(|G-T|-2)!|T|!}{(|G|-1)!} [\mu(T \cup \{i,j\}) - \mu(T \cup \{i\})) - \mu(T \cup \{j\}) + \mu(T)]. \tag{1.10}$$

In case of 2-additive capacities, the Shapley value (1.9) and the interaction index (1.10) can be reformulated as follows:

$$\varphi(\{i\}) = a(\{i\}) + \sum_{j \in G \setminus \{i\}} \frac{a(\{i,j\})}{2}, \ \ i \in G, \tag{1.11}$$

$$\varphi(\{i,j\}) = a(\{i,j\})\,. \tag{1.12}$$

1.3 The Analytic Hierarchy Process

The Analytic Hierarchy Process (AHP) is an MCDA method helping to build ratio scales for measuring performance on considered criteria and importance of the same criteria. The problem at hand is structured by AHP in a hierarchical way, as shown in Figure 1.1, where the overall goal is put at the top of the hierarchy, the alternatives being the object of the decision are placed at the bottom of the hierarchy while, the criteria on which the alternatives need to be evaluated are put in the middle of the hierarchy between the overall goal and the alternatives themselves.

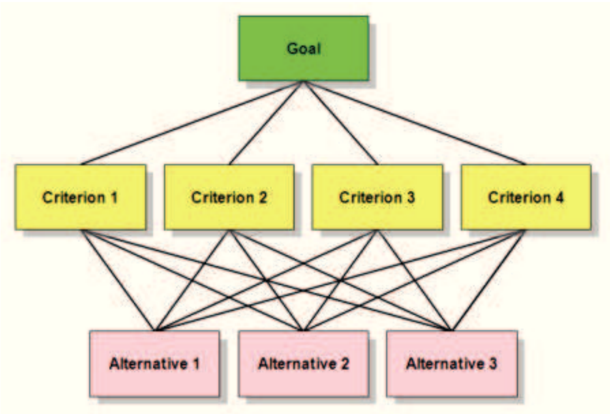


Figure 1.1: AHP structure

Given the n criteria $g_1, \ldots, g_n$ and their priorities $w_1, \ldots, w_n$, let us consider the matrix

$$M = \begin{pmatrix} w_1/w_1 & w_1/w_2 & \cdots & w_1/w_n \\ w_2/w_1 & w_2/w_2 & \cdots & w_2/w_n \\ \vdots & \vdots & \vdots & \vdots \\ w_n/w_1 & w_n/w_2 & \cdots & w_n/w_n \end{pmatrix}$$

where each entry $m_{ij} = w_i/w_j$ is the ratio between the priorities w_i and w_j. Of course, the matrix M is such that $m_{ij} = 1/m_{ji}$ for all i, j (M is therefore reciprocal) and $m_{ij} = m_{ik} \cdots m_{kj}$ for all i, j, k (M is therefore consistent). Moreover, $m_{ii} = 1$ for all $i = 1, \ldots, n$. Multiplying on the right the matrix M for the weight vector $w = (w_1, \ldots, w_n)^T$, we get

$$\begin{pmatrix} w_1/w_1 & w_1/w_2 & \cdots & w_1/w_n \\ w_2/w_1 & w_2/w_2 & \cdots & w_2/w_n \\ \vdots & \vdots & \vdots & \vdots \\ w_n/w_1 & w_n/w_2 & \cdots & w_n/w_n \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = n \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$

or, in a compact form,

$$Mw = nw. \tag{1.13}$$

Let us notice from equation (1.13) that n is an eigenvalue of the matrix M, $rank(M) = 1$ (since each row is a multiple of the first row) and, consequently, n is the only eigenvalue of M different from zero. The priorities vector w is, therefore, the eigenvector associated to the eigenvalue n.

The DM is not able to provide exact numbers for the priorities $w_1, \ldots, w_n$, but he is able to perform pairwise judgments on the considered criteria. For this reason, in AHP, the DM is asked to compare each pair of criteria $\{g_i, g_j\}$ expressing a verbal judgment such as g_i is "as important as " g_j or, g_i "moderately dominates", "strongly dominates", "very strongly dominates" or "extremely dominates" g_j. These judgments are numerically coded with 1, 3, 5, 7 and 9 while the values 2, 4, 6 and 8 express a compromise between the previous values. Denoting by a_{ij} the pairwise comparison between the criteria g_i and g_j, it is possible to build a positive square reciprocal matrix of order n:

$$M' = \begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 1/a_{12} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 1/a_{1n} & 1/a_{2n} & \cdots & 1 \end{pmatrix} \tag{1.14}$$

Denoting by λ_{max} the maximal eigenvalue of M', in [146] it is proved that $\lambda_{max} \geq n$ and that M' is consistent iff $\lambda_{max} = n$. For this reason, in order to check the consistency of the comparisons provided by the DM, AHP considers the consistency index $CI = (\lambda_{max} - n)/(n-1)$ and the consistency ratio $CR = CI/RI$, where RI is the consistency index of 500 reciprocal square matrices of order n filled in a random way. If $CR \leq 0.1$, then the judgments provided by the DM are consistent enough and the eigenvector associated to the eigenvalue λ_{max} is an accepted approximation of the priority vector w.

Once an estimate $(w_1, \ldots, w_n)$ of the priority vector of weights has been obtained, the DM is therefore asked to compare pairwise the alternatives stating how much better is an alternative than another with respect to each considered criterion. Again, AHP is applied to obtain the priorities of each alternative on each criterion. Formally, denoting by $M^{(i)}$ the pairwise comparison matrix of order $|A|$ related to criterion g_i for all $i = 1, \ldots, n$, and by $p^{(i)} = (p_1^{(i)}, \ldots, p_{|A|}^{(i)})$ the priority vectors obtained by matrices $M^{(i)}$, the final evaluation $U(a_l)$ of the alternative $a_l \in A$ is obtained as

$$U(a_l) = \sum_{i=1}^{n} p_l^i w_i.$$

1.4 Robust Ordinal Regression

Because the application of each decision model involves the definition of the values of several parameters (the marginal value functions in MAVT, the capacity in the case of the Choquet integral preference model or the thresholds, weights and cutting levels in outranking methods), in literature a direct and an indirect techniques are used. With the direct technique, the DM provides directly values of the parameters involved in the model at hand. Therefore, he is able to define the shape of the considered value functions in MAVT or the Möbius coefficients necessary to the application of the Choquet integral preference model or the thresholds in the outranking methods. With the indirect technique, the DM provides some preference information on reference alternatives $A^R \subseteq A$ in terms of pairwise comparisons, or on considered criteria in terms of comparison between their importance, from which a set of parameters compatible with these preferences can be inferred. In the following we shall call *compatible model* a set of parameters compatible with the preference information provided by the DM. Since the direct preference information is more demanding from the cognitive point of view for the DM, the indirect preference information is preferred in practice. The indirect preference information is used in the ordinal regression paradigm. The ordinal regression paradigm has been applied within the approaches using a value function as preference model [29, 97, 98, 133, 160], and those using an outranking relation as preference model [126, 127].

In general, more than one compatible model exists. Each of them provides the same recommendations on the reference alternatives but they can provide different recommendations on the other alternatives that do not belong to the reference set. For example, let us consider two alternatives $a^*, b^* \in A^R$ for which the DM stated that a^* is preferred to b^*. Then, all compatible models will be concordant with the preference of a^* over b^* but, considering other two alternatives $a, b \in A$ on which the DM did not provide any information (in the sense that the DM did not state neither that a is preferred to b nor that b is preferred to a), it is possible that for some compatible models, a is preferred to b, while for other compatible models, b is preferred to a. For this reason, the choice of

only one among the many compatible models is, in some sense, arbitrary and meaningless.
Robust Ordinal Regression (ROR) [35, 36, 86] copes with this problem taking into account simultaneously all models compatible with the preferences provided by the DM. To do this, ROR builds two preference relations, one necessary and one possible. The couple (a, b) belongs to the necessary preference relation iff a is at least as good as b for all compatible models, while the couple (a, b) belongs to the possible preference relation iff a is at least as good as b for at least one model compatible with the preferences provided by the DM (for a discussion on the axiomatic basis of the necessary and possible preference relations see [64]). ROR has been already applied to value functions in [37, 86, 87, 88, 103], to the Choquet integral preference model in [8], to outranking methods in [38, 78, 103] and to interactive evolutionary multiobjective optimization in [24].

Even if the recommendations obtained by ROR are more robust than the recommendations got by applying one compatible model only, in some cases, one needs to assign a single number to each alternative being representative of its goodness with respect to the problem at hand. In these cases, one needs to consider only one among the plurality of models compatible with the preferences provided by the DM and, therefore, the most representative model has been introduced in [53]. The most representative model is the compatible model summarizing the results of the ROR. In particular, it maximizes the difference between the alternatives (a, b) such that a is necessarily preferred to b and b is not necessarily preferred to a and, at the same time, it minimizes the difference between the alternatives $a, b \in A$ such that neither a is necessarily preferred to b nor b is necessarily preferred to a. The most representative model has been computed for value functions [79, 101, 102], as well as for the Choquet integral preference model [7] and for outranking methods [104].

1.5 SMAA methodology

Even if in a different way with respect to ROR, Stochastic Multiobjective Acceptability Analysis (SMAA) methodologies are based on the indirect preference information provided by the DM and aim to explore the whole set of models compatible with this preference. SMAA methodologies are, in general, applied to problems where the uncertainty is so significant that it should be considered explicitly. Incomplete information on considered criteria as well as on performances of the different alternatives on the criteria at hand are represented by suitable probability distributions $f_W(w)$ and $f_\chi(\xi)$ where W is the set of parameters representing the preferences of the DM, $\chi \subseteq \mathbb{R}^{|A|\times n}$ is the set of all possible performances matrices ξ of the alternatives at hand and, in particular,

$$\xi = [\xi_{ki}]_{ki} = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_{|A|} \end{bmatrix} \in \chi.$$

Different variants of methods have been proposed under the SMAA framework depending on the underlying model used to represent the preferences provided by the DM (for a survey on SMAA methods see [164]). In the following, we describe the indices used by the SMAA methods just in case the used preference model is the simple weighted sum:

$$U(\xi_k, w) = \sum_{i=1}^{n} w_i \xi_{ki}$$

where $\xi_{ki} = g_i(a_k)$ is the performance of the alternative a_k on criterion g_i, chosen the performance matrix $\xi \in \chi$.

As already mentioned in the previous sections, the application of a value function (in this case of a weighted sum), permits to get a complete ranking of the alternatives at hand. For this reason, given an alternative $a_k \in A$, a vector of weights $w \in W = \{w \in \mathbb{R}^n : w_j \geq 0, \forall j, \text{ and } \sum_{j=1}^{n} w_j = 1\}$ and a performance matrix $\xi \in \chi$, SMAA computes:

- The position got by a_k in the final ranking of the population

$$rank(k, \xi, w) = 1 + \sum_{r \neq k} \rho(U(\xi_r, w) > U(\xi_k, w)),$$

  where $\rho(true) = 1$ and $\rho(false) = 0$

- The set of favourable rank weights $W_k^r(\xi)$, with $r = 1, \ldots, |A|$, that is the set of weights giving to a_k the r-th position in the ranking of the alternatives

$$W_k^r(\xi) = \{w \in W : rank(k, \xi, w) = r\}$$

Based on $W_k^r(\xi)$, the main results of SMAA analysis are the rank acceptability indices, the central weight vectors, the confidence factors for each alternative and the pairwise winning indices for each couple of alternatives $a_k, a_r \in A$.

- The rank acceptability index

$$b_k^r = \int_{\xi \in \chi} f_X(\xi) \int_{w \in W_k^r(\xi)} f_W(w) \; dw \; d\xi$$

is the frequency with which an alternative fills the position r in the obtained final ranking; obviously, the greater the value of b_k^1 the better a_k is and, vice versa, the greater the value of $b_k^{|A|}$, the worse alternative a_k. Let us remember that in the first version of the SMAA methodology [113], only b_k^1 was computed for each alternative and it was called *acceptability index*;

- The central weight vector

$$w_k^c = \frac{1}{b_k^1} \int_{\xi \in \chi} f_X(\xi) \int_{w \in W_k^1(\xi)} f_W(w) w \; dw \; d\xi$$

describes the preferences of a typical DM that make alternative a_k the most preferred;

- The confidence factor

$$p_k^c = \int_{\substack{\xi \in \chi: \; U(\xi_k, w_k^c) \geq U(\xi_r, w_k^c) \\ \forall r = 1, \ldots, |A|}} f_X(\xi) \; d\xi$$

measures if the criteria measurements are accurate enough to discern the efficient alternatives;

- The pairwise winning index [116, 165]

$$p(a_k, a_r) = \int_{w \in W} f_W(w) \int_{\xi \in \chi: \; U(\xi_k, w) \geq U(\xi_r, w)} f_X(\xi) \; d\xi \; dw$$

is the frequency with which an alternative a_k is at least as good as an alternative a_r.

Since the computation of all mentioned indices involves to solve multidimensional integrals, approximations of these integrals are computed via Monte Carlo simulations.

Chapter 2

Contributions related to the Multiple Criteria Hierarchy Process

As stated in the introductory section, all MCDA methods assume that the evaluation criteria are sited at the same level even if, in general, this is not true. Many examples of problems presenting an hierarchy of criteria can be presented. Let us suppose, for example, that a committee has to evaluate the feasibility of different projects. Of course, it has to take into account economic, environmental and social aspects and, going more in depth, under each of these aspects several criteria can be considered. Total amount of the investment and possible earning can be considered as subcriteria of the economic macrocriterion, soil sustainability and water sustainability can be listed as subcriteria of the environmental macrocriterion while, number of inhabitants living in the interested area and means of transport (bus, metro, train) connecting the area to the city center can be considered under the social aspects.

The Multiple Criteria Hierarchy Process (MCHP) has been recently proposed to deal with decision making problems in which criteria are not all at the same level, but they are structured in a hierarchical way as in the example provided above. The advantage of the MCHP is that it permits to decompose and to make easier the preference elicitation. Indeed, the basic idea of the MCHP relies on consideration of preference relations at each node of the hierarchy tree of criteria. These preference relations concern both the phase of eliciting preference information, and the phase of analyzing a final recommendation by the DM. On one hand, the DM can provide partial preference information related only to a particular aspect of the problem avoiding to take into account the whole set of criteria simultaneously. In the example provided above, we can assume that the committee is composed of three members, whose one is an expert in economy, one in environmental aspects and one in social aspects. Of course, each of them will be more confident in providing information on his field of expertise while he could be more in difficulty in giving information on other fields or, even more, at a global level taking into account the all the criteria together. On the other hand, the use of the MCHP permits to get as output more precise recommendations not only at a global level but considering each particular aspect of the problem at hand. For example, even if a global ranking of the considered projects is obviously necessary to decide which of them deserves to be financed, more partial information on the three aspects at hand can be very relevant from the side of the people who should invest in the project. Indeed, considering a project a evaluated as "medium" at comprehensive level, it is possible that it is evaluated as "good" with respect to environmental and economic aspects and "bad" with respect to social aspects. Such a type of output information can be obtained by the application of the MCHP. The MCHP has already been applied in the case in which the preference model is a value function [37] or an outranking method [38]. In the thesis we went further, applying the MCHP to the sorting method UTADIS [43] and to the Choquet integral preference model [31].

In the first contribution, the application of the MCHP to the UTADIS method permits to assign each alternative to one or more of the preferentially ordered classes not only at a comprehensive way but also considering a particular subcriterion in the hierarchy of criteria. This extension regards both the direct and the indirect preference information and, in the case of the indirect preference information, it applies ROR computing the necessary and possible assignments at each node of the hierarchy of criteria.

In the second contribution, the application of the MCHP to the Choquet integral preference model permits, instead, to take into account the possible positive and negative interactions that can be observed between criteria organized in a hierarchical way. Also in this case, the proposed extension regards the direct and the indirect preference information and, in particular, in the indirect case we applied both ROR and SMAA to take into account the plurality of models compatible with the preferences provided by the DM.

The applications of the MCHP to the UTADIS method and to the Choquet integral preference model are presented in sections 2.1 and 2.2, respectively.

2.1 Multiple Criteria Hierarchy Process for Sorting Problems Based on Ordinal Regression with Additive Value Functions

2.1.1 Introduction

In many decision making problems, decisions concerning a set of alternatives are based on different evaluation criteria organized in a hierarchical structure. Such a hierarchy introduces a decomposition of the primary objective into separate dimensions, which are then further analyzed in sub-dimensions, up to the lowest level of the hierarchy, which consists of the elementary criteria. Structuring decision problems following such a hierarchical scheme is particularly useful in situations that require consideration of large sets of criteria describing different aspects of the problem at hand. Dealing with complex families of criteria of diverse nature, poses significant cognitive burden to decision makers (DMs). Thus, using a hierarchical decomposition facilitates the analysis as it allows DMs to deal with more manageable elementary dimensions. Furthermore, working with such a hierarchy provides detailed insights on all partial dimensions of the problem, instead of focusing solely on the comprehensive level.

A common approach to deal with hierarchies of criteria in MCDA is the analytic hierarchy process [148], but its fundamental problems are well-documented in the literature (see, for example, [10]). Recently, the Multiple Criteria Hierarchy Process (MCHP) has been introduced as an alternative [5, 37, 38]. The MCHP introduces a new modeling framework that allows the construction of sound decision models in decision problems with a hierarchical structure, through MCDA techniques based on the preference disaggregation paradigm [98]. The MCHP is able to take into account preference information not only at a comprehensive level but also at all lower levels of the hierarchy, and provide recommendations in a similar form.

In previous studies, the MCHP has been introduced in the context of choice and ranking problems, where the objective is either to choose the best alternative(s) among those considered (choice) or to rank-order the alternatives from the best to the worst ones. In these contexts, the MCHP has been employed to construct decision models with outranking methods such as ELECTRE and PROMETHEE [38], value function models [37], as well as with the Choquet integral preference model [5].

In this study, we extend the MCHP framework to multiple criteria sorting (classification) problems, where the objective is to assign a set of alternatives to predefined (ordinal) decision classes. Such problems often arise in many domains [180] and they have attracted much interest in MCDA over the past decade. Sorting problems have been dealt in the literature using outranking relations (e.g., the ELECTRE Tri method [178]), value functions, and decision rules [81, 82, 83, 155]. In this paper, we focus on value function models, which constitute a convenient and easy way of modeling DMs preferences in MCDA problems. The best-known method based on this modeling approach for multiple criteria sorting problems is the UTADIS method (UTilités Additives DIScriminantes) and its variants [43, 45, 179]. In this paper, we extend the UTADIS method to problems having a hierarchical structure by applying the MCHP framework. In order to reduce the cognitive effort of the DM, we also extend the UTADIS^{GMS} method [87], putting it in the MCHP framework. UTADIS^{GMS} is the generalization of UTADIS to the Robust Ordinal Regression (ROR) setting [35, 36, 86]. ROR is a family of methods taking into account not only one but all instances of an assumed preference model being compatible with the preference information provided by the DM. In that regard, this study contributes to the literature on multiple criteria sorting through the extension of existing techniques for inferring decision models from sorting decision examples, using a formal framework of MCHP, which allows the input preference information to be decomposed into smaller and more manageable aspects of the problem. In order to illustrate the proposed methodology, we employ a case study involving a financial decision problem, namely the performance rating of banks. In a supervisory context, bank rating is a complex process that requires the consideration of all aspects of bank operation, financial status, and risk profile. This case study fits well the framework of MCHP and multiple criteria sorting, and thus, it illustrates well the potentials of the proposed modeling approach in practice.

The rest of the paper is organized as follows: In the next section, a general problem setting is provided. Section 2.1.3 describes the MCHP extension of the UTADIS method to decision problems with a hierarchical structure, while in section 2.1.4 the integration of MCHP and UTADIS^{GMS} is explained in detail. The application to bank performance evaluation is presented in section 2.1.5. Finally, section 2.1.6 concludes the paper and provides some future research directions.

2.1.2 General Setting

A set of alternatives $A = \{a, b, \ldots\}$ is evaluated on a set of criteria structured in a hierarchical way in l different levels. The complete set of criteria (from all levels) will be denoted by $\mathcal{G}$, while the set of indices of criteria will be denoted by $\mathcal{I}_{\mathcal{G}}$. The criteria at the lowest level of the hierarchy will be called elementary criteria and the alternatives will be directly evaluated on these criteria only. The set of indices of elementary criteria will be denoted by EL, while the set of indices of elementary criteria descending from node $G_{\mathbf{r}}$ of the hierarchy ($\mathbf{r} \in \mathcal{I}_{\mathcal{G}}$), will be denoted by $E(G_{\mathbf{r}})$. Each node of the hierarchy represents a particular sub-dimension of the problem, with $G_{\mathbf{0}}$ corresponding to the root of the hierarchy (i.e., $G_{\mathbf{0}} = \mathcal{G}$). Without loss of generality, we shall suppose that all elementary criteria are to be maximized (i.e., preference increases with the value of each criterion).

Furthermore, by $n(\mathbf{r})$ we shall denote the number of criteria $G_{(\mathbf{r},1)}, \ldots, G_{(\mathbf{r},n(\mathbf{r}))}$ descending from $G_{\mathbf{r}}$ in the next (lower) level of the hierarchy. Obviously, the elementary criteria are not further decomposed into subcriteria. By LBO we shall denote the indices of the criteria from the next to the last level of the hierarchy, while $LB(G_{\mathbf{r}})$ will denote the set of indices for criteria descending from $G_{\mathbf{r}}$ and located at the next to the last level.

Assuming that the set of elementary criteria is mutually preferentially independent [109, 174], their aggregation is possible with an additive value function $U : A \rightarrow [0, 1]$, such that:

$$U(a) = \sum_{\mathbf{t} \in EL} u_{\mathbf{t}}(g_{\mathbf{t}}(a))$$

where $u_{\mathbf{t}}$ are marginal value functions related to elementary criteria $g_{\mathbf{t}}$.

In the MCHP context, assuming that at each level criteria are preferentially independent, it is possible to consider a partial value function for each (non-elementary) criterion $G_{\mathbf{r}}$, $\mathbf{r} \in \mathcal{I}_{\mathcal{G}}$ as follows:

$$U_{\mathbf{r}}(a) = \sum_{\mathbf{t} \in E(G_{\mathbf{r}})} u_{\mathbf{t}}(g_{\mathbf{t}}(a)).$$

An obvious consequence is that

$$U_{\mathbf{r}}(a) = \sum_{j=1}^{n(\mathbf{r})} U_{(\mathbf{r},j)}(a) \tag{2.1}$$

where $U_{(\mathbf{r},j)}(a)$ represents the value of alternative a according to the j-th subcriterion of $G_{\mathbf{r}}$, for all $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$ (more details on MCHP can be found in [37]).

For each criterion $G_{\mathbf{r}}$ above the level of elementary criteria, the sorting procedure with respect to subcriteria descending directly from $G_{\mathbf{r}}$ consists in assigning each alternative from A to one among $p_{\mathbf{r}}$ decision classes $C_1, \ldots, C_{p_{\mathbf{r}}}$, where $C_{p_{\mathbf{r}}}$ is the class of top performing alternatives and C_1 is the class of the worst alternatives. Note that the sorting with respect to subcriteria descending directly from different criteria $G_{\mathbf{r}}$ could involve different values of $p_{\mathbf{r}}$, i.e., the number of classes to which an alternative can be assigned could depend on $G_{\mathbf{r}}$. For each criterion $G_{\mathbf{r}}$ above the elementary level, class C_h ($h \in \{1,\ldots,p_{\mathbf{r}}\}$) is defined by lower and upper value thresholds $b^{\mathbf{r}}_{h-1}$ and $b^{\mathbf{r}}_{h}$, such that $b^{\mathbf{r}}_{h-1} < b^{\mathbf{r}}_{h}$, defined on the value function scale. It follows that $0 = b^{\mathbf{r}}_0 < b^{\mathbf{r}}_1 < \ldots < b^{\mathbf{r}}_{p_{\mathbf{r}}-1} < b^{\mathbf{r}}_{p_{\mathbf{r}}} = \sum\limits_{\mathbf{t}\in E(G_{\mathbf{r}})} u_{\mathbf{t}}(x^{m_{\mathbf{t}}}_{\mathbf{t}})$, where the value of $b^{\mathbf{r}}_{p_{\mathbf{r}}}$ is the maximum level of the value function for $G_{\mathbf{r}}$ (with $x^{m_{\mathbf{t}}}_{\mathbf{t}}$ being the best performance on elementary criterion $g_{\mathbf{t}}$ over all alternatives from A).

2.1.3 MCHP and the UTADIS method

Consider an assignment of alternative $a \in A$ to class C_h ($h \in \{1,...,p_{\mathbf{r}}\}$) with respect to subcriteria descending directly from criterion $G_{\mathbf{r}}$ ($\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$). In the following, instead of criterion $G_{\mathbf{r}}$, we shall often use the term *node* $G_{\mathbf{r}}$, in order to stress that the assignment takes place in a particular place of the hierarchy tree.

Moreover, the assignment of alternatives with respect to subcriteria descending directly from criterion $G_{\mathbf{r}}$ ($\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$) will be called the *assignment in node* $G_{\mathbf{r}}$.

Definition 2.1.1. *In node* $G_{\mathbf{r}}$ ($\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$)*, alternative* a *is assigned to class* C_h ($h = 1,\ldots,p_{\mathbf{r}}$) $\left(\text{denoted as } a \underset{\mathbf{r}}{\rightarrow} C_h\right)$*, iff* $b^{\mathbf{r}}_{h-1} \leq U_{\mathbf{r}}(a) < b^{\mathbf{r}}_{h}$.
As a consequence, in node $G_{\mathbf{r}}$*,*

- *a is assigned to at least class* C_h $\left(a \underset{\mathbf{r}}{\rightarrow} C_{\geq h}\right)$*, iff* $U_{\mathbf{r}}(a) \geq b^{\mathbf{r}}_{h-1}$*,*
- *a is assigned to at most class* C_h $\left(a \underset{\mathbf{r}}{\rightarrow} C_{\leq h}\right)$*, iff* $U_{\mathbf{r}}(a) < b^{\mathbf{r}}_{h}$ *(the inequality becomes weak if* $h = p_{\mathbf{r}}$*, that is* $U_{\mathbf{r}}(a) \leq b^{\mathbf{r}}_{p_{\mathbf{r}}}$*),*
- *a is assigned to some class in the interval* $[C_{h_1}, C_{h_2}]$ $(1 < h_1 < h_2 < p_{\mathbf{r}})$ $\left(a \underset{\mathbf{r}}{\rightarrow} [C_{h_1}, C_{h_2}]\right)$*, iff* $b^{\mathbf{r}}_{h_1-1} \leq U_{\mathbf{r}}(a) < b^{\mathbf{r}}_{h_2}$.

In what follows, in order to simplify the presentation and without loss of generality, we assume that the same classes apply in all nodes of the hierarchy tree. This means that the number of classes $p_{\mathbf{r}}$ to which each alternative can be assigned does not depend on the considered node $G_{\mathbf{r}}$. Consequently, $p_{\mathbf{r}} = p$ for all $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$.

A first desirable coherence property for hierarchical multiple criteria sorting methods is the following. If an alternative $a \in A$ is assigned to class C_h in all nodes directly descending from $G_{\mathbf{r}}$, then it should also be assigned to the same class in node $G_{\mathbf{r}}$. For example, if student S is assigned to the class of good students in the nodes corresponding to Algebra and Analysis, being the only two subcriteria of the criterion Mathematics, then S has to be assigned to the class of good students also in the node of Mathematics. Henceforth, we shall refer to this property as the *first coherence property of hierarchical multiple criteria sorting.*

A second desirable coherence property for hierarchical multiple criteria sorting methods is the following. If an alternative $a \in A$ is assigned to at least class C_h, i.e., to class C_h or better, in all nodes directly descending from $G_{\mathbf{r}}$, then it should also be assigned to at least class C_h in node $G_{\mathbf{r}}$. Coming back to the previous example, if student S is assigned to at least medium class of students in the nodes corresponding to Algebra and Analysis, then S has also to be assigned to at least the medium class of students in the node of Mathematics. Let us call this property *second coherence property of hierarchical multiple criteria sorting.* Of course, another coherence property for hierarchical multiple criteria sorting methods is symmetric to the second property, i.e., if an alternative $a \in A$ is assigned to at most class C_h (to class C_h or worse), in all nodes directly descending from $G_{\mathbf{r}}$, then it should also be assigned to at most class C_h in node $G_{\mathbf{r}}$. Henceforth, this property will be referred to as the *third coherence property of hierarchical multiple criteria sorting.* The second and third coherence properties of hierarchical multiple criteria sorting can be synthesized as follows: in node $G_{\mathbf{r}}$, an alternative $a \in A$ should be assigned to an interval of contiguous classes, included in the interval of classes having as extrema the worst and the best classes to which a is assigned in nodes directly descending from $G_{\mathbf{r}}$. For example, if student S is assigned to the interval of classes from moderate to relatively good students in the node of Algebra, and to the interval of classes from medium to good students in the node of Analysis, then student S has to be assigned to an interval of classes from moderate to good students in the node of Mathematics. Even if this coherence property is the mere synthesis of the above second and third coherence properties, we shall refer to it as the *fourth coherence property for hierarchical multiple criteria sorting methods*, because it will be useful to recall it in the subsequent discussion.

Proposition 2.1.1 given below says that the first and the fourth coherence properties for hierarchical multiple criteria sorting methods coincide, and that they hold if and only if the value thresholds separating the classes in node $G_{\mathbf{r}}$ are equal to the sum of the corresponding value thresholds separating the classes in the nodes directly descending from $G_{\mathbf{r}}$ (see the Appendix for the proofs). Indeed, this condition is expressed as statement 1 of Proposition 2.1.1, whereas the first and the fourth coherence properties for hierarchical multiple criteria sorting methods correspond to statements 2 and 3, respectively.

Proposition 2.1.1. *The three following statements are equivalent:*

1. *In each node* $G_{\mathbf{r}}$, $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, $b_h^{\mathbf{r}} = \displaystyle\sum_{j=1}^{n(\mathbf{r})} b_h^{(\mathbf{r},j)}$ *for all* $h = 0, \dots, p$,

2. *In each node $G_{\mathbf{r}}$, $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, if $a \xrightarrow[(\mathbf{r},j)]{} [C_{h_j}, C_{k_j}]$ for all $j = 1, \ldots, n(\mathbf{r})$, then $a \underset{\mathbf{r}}{\rightarrow} [C_h, C_k]$ where $h = \min\limits_{j=1,\ldots,n(\mathbf{r})} h_j$, and $k = \max\limits_{j=1,\ldots,n(\mathbf{r})} k_j$,*

3. *In each node $G_{\mathbf{r}}$, $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, if $a \xrightarrow[(\mathbf{r},j)]{} C_h$ for all $j = 1, \ldots, n(\mathbf{r})$ then $a \underset{\mathbf{r}}{\rightarrow} C_h$.*

Since we would like our hierarchical sorting approach to respect points 2 and 3 of Proposition 2.1.1, we shall assume that $b^{\mathbf{r}}_h = \sum\limits_{j=1}^{n(\mathbf{r})} b^{(\mathbf{r},j)}_h$ in each node $G_{\mathbf{r}}, \mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$. As a consequence of this choice, it is sufficient to define the value thresholds in nodes from the last but one level of the hierarchy, because for any other higher level node $G_{\mathbf{r}}$, $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus \{EL \cup LBO\}$ it holds that

$$b^{\mathbf{r}}_j = \sum_{\mathbf{s} \in LB(G_{\mathbf{r}})} b^{\mathbf{s}}_j, \text{ for all } j = 0, \ldots, p.$$

In order to construct the additive value function and define the value thresholds, one can use a direct or an indirect approach. In the former case, the DM is asked to specify explicitly the parameters of the model (value thresholds in this case), following an direct assessment protocol designed specifically for the type of model under consideration (for example of such a protocol for additive value models see [18]). On the other hand, in an indirect approach [98] the DM is asked to provide some comprehensive preference information on the assignment of some reference alternatives (i.e., taking into account the full set of criteria present in the hierarchy) and/or partial preference information (i.e., considering a particular dimension of the problem, corresponding to criterion $G_{\mathbf{r}}$, being an intermediate node in the hierarchy tree). With such preference information at hand, it is possible to infer values for the parameters of the model that are compatible with the judgments provided by the DM. This can be achieved considering the following set of constraints (in accordance with [87], henceforth (U, b) will be used to denote a value function and a set of value thresholds compatible with the preferences of the DM, whereas $\mathcal{U}$ will denote the set of all compatible instances of this model):

$$\left.\begin{array}{l}\left.\begin{array}{l}U_{\mathbf{r}}(a)\geq b^{\mathbf{r}}_{h-1},\\[2mm] U_{\mathbf{r}}(a)-b^{\mathbf{r}}_{h}\leq-\varepsilon\end{array}\right\}\text{if } a\underset{\mathbf{r}}{\rightarrow}C_h\\[2mm]
U_{\mathbf{r}}(a)\geq b^{\mathbf{r}}_{h-1},\ \text{if } a\underset{\mathbf{r}}{\rightarrow}C_{\geq h}\\[2mm]
U_{\mathbf{r}}(a)-b^{\mathbf{r}}_{h}\leq-\varepsilon,\ \text{if } a\underset{\mathbf{r}}{\rightarrow}C_{\leq h}\\[2mm]
\left.\begin{array}{l}U_{\mathbf{r}}(a)\geq b^{\mathbf{r}}_{h_1-1},\\[2mm] U_{\mathbf{r}}(a)-b^{\mathbf{r}}_{h_2}\leq-\varepsilon\end{array}\right\}\text{if } a\underset{\mathbf{r}}{\rightarrow}[C_{h_1},C_{h_2}]\\[2mm]
u_{\mathbf{t}}(x^k_{\mathbf{t}})\geq u_{\mathbf{t}}(x^{k-1}_{\mathbf{t}}),\ \ k=1,\ldots,m_{\mathbf{t}},\ \text{for all } \mathbf{t}\in EL,\\[2mm]
u_{\mathbf{t}}(x^0_{\mathbf{t}})=0,\ \text{for all } \mathbf{t}\in EL,\ \ \text{and} \displaystyle\sum_{\mathbf{t}\in EL}u_{\mathbf{t}}(x^{m_{\mathbf{t}}}_{\mathbf{t}})=1\\[2mm]
b^{\mathbf{s}}_{h}\geq b^{\mathbf{s}}_{h-1}+\varepsilon, h=1,\ldots,p,\ \ \text{for all } \mathbf{s}\in LBO,\\[2mm]
b^{\mathbf{s}}_{0}=0,\ \text{and } b^{\mathbf{s}}_{p}=\displaystyle\sum_{\mathbf{t}\in E(G_{\mathbf{s}})}u_{\mathbf{t}}(x^{m_{\mathbf{t}}}_{\mathbf{t}}),\ \ \text{for all } \mathbf{s}\in LBO,\\[2mm]
b^{\mathbf{r}}_{h}=\displaystyle\sum_{\mathbf{s}\in LB(G_{\mathbf{r}})}b^{\mathbf{s}}_{h}, \text{for all } h=0,\ldots,p,\ \ \text{and for all } \ \mathbf{r}\in\mathcal{I}_{\mathcal{G}}\setminus\{EL\cup LBO\}
\end{array}\right\}E^{A^R}$$

where $x^k_{\mathbf{t}}$, $k=0,\ldots,m_{\mathbf{t}}$, are the $m_{\mathbf{t}}+1$ different performances on elementary criterion $g_{\mathbf{t}}$ attained by alternatives in A (arranged in ascending order); $x^0_{\mathbf{t}}$ and $x^{m_{\mathbf{t}}}_{\mathbf{t}}$ are, respectively, the worst and the best performances of alternatives on elementary criterion $g_{\mathbf{t}}$, while ε is an auxiliary variable used to translate the strict inequality constraints to weak inequality constraints.

If E^{A^R} is feasible and $\varepsilon^*>0$, where $\varepsilon^*=\max\varepsilon$ subject to E^{A^R}, then there exists at least one instance (U,b) compatible with the preferences provided by the DM. The readers interested to conditions ensuring the existence of an additive representation of ordered partitions could look at [18].

Remark 2.1.1. *Let us observe that if the number of classes considered in each node $G_{\mathbf{r}}$ is different (different values of $p_{\mathbf{r}}$ for all $G_{\mathbf{r}}$), then the direct and the indirect approaches explained above remain valid. In particular, in the indirect approach, one has to consider the set of constraints $E^{A^R}_1$ obtained from E^{A^R} by replacing the last three constraints with the following ones:*

$$b^{\mathbf{r}}_{h}\geq b^{\mathbf{r}}_{h-1}+\varepsilon, h=1,\ldots,p_{\mathbf{r}},\ \ \textit{for all}\, \mathbf{r}\in\mathcal{I}_{\mathcal{G}}\setminus EL, \tag{2.2}$$

$$b^{\mathbf{r}}_{0}=0,\ \textit{and}\, b^{\mathbf{r}}_{p_{\mathbf{r}}}=\sum_{\mathbf{t}\in E(G_{\mathbf{r}})}u_{\mathbf{t}}(x^{m_{\mathbf{t}}}_{\mathbf{t}}),\ \ \textit{for all}\, \mathbf{r}\in\mathcal{I}_{\mathcal{G}}\setminus EL. \tag{2.3}$$

The last constraint in E^{A^R} does not hold anymore (except for the case $h=0$ and $h=p_{\mathbf{r}}$ in consequence *of eq. (2.3)), since the number and meaning of the different thresholds obviously depend on the number of classes to which each alternative can be assigned in node* $G_{\mathbf{r}}$. *For example, let us consider a small hierarchy in which a root criterion* $G_{\mathbf{0}}$ *has subcriteria* $G_{\mathbf{1}}$ *and* $G_{\mathbf{2}}$*, and each alternative can be assigned to two classes in node* $G_{\mathbf{0}}$ ($p_{\mathbf{0}} = 2$) *while it can be assigned to three and four classes in nodes* $G_{\mathbf{1}}$ *and* $G_{\mathbf{2}}$*, respectively* ($p_{\mathbf{1}} = 3$ *and* $p_{\mathbf{2}} = 4$)*. In this case, there will be three value thresholds in node* $G_{\mathbf{0}}$ ($\{b_0^0, b_1^0, b_2^0\}$)*, four value thresholds in node* $G_{\mathbf{1}}$ ($\{b_0^1, b_1^1, b_2^1, b_3^1\}$)*, and five value thresholds in node* $G_{\mathbf{2}}$ ($\{b_0^2, b_1^2, b_2^2, b_3^2, b_4^2\}$).

2.1.4 MCHP and the UTADIS^{GMS} method

In general, more than one instance of the preference model could be compatible with the preference information provided by the DM. Each of these instances restores the given information in the same way, but each one of them could provide different recommendations on alternatives outside the reference set. In this case, choosing a single compatible instance of the preference model may lead to a loss of possibly important information. For this reason, Robust Ordinal Regression (ROR) [35, 36, 86] takes into account the whole set of instances of the preference model compatible with the preference information provided by the DM, by building *necessary* and *possible* preference relations that hold for all or for at least one compatible instance of the preference model.

In the MCHP context, the DM could be therefore interested to know not only to which class an alternative could be necessarily or possibly assigned taking into account the whole set of criteria, but also to which class it could be necessarily or possibly assigned with respect to a criterion corresponding to a particular node of the hierarchy tree.

In this section, we extend UTADIS^{GMS} [87] to the MCHP context, by reformulating the definition of the necessary and possible assignments as follows:

Definition 2.1.2. *In any node* $G_{\mathbf{r}}$, ($\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$) *in the hierarchy tree,*

- $a \in A$ *is necessarily assigned to at least class* C_h*, denoted by* $a \xrightarrow[\mathbf{r}]{N} C_{\geq h}$*, iff* $U_{\mathbf{r}}(a) \geq b_{h-1}^{\mathbf{r}}$ *for all compatible* (U, b)*,*
- $a \in A$ *is possibly assigned to at least class* C_h*, denoted by* $a \xrightarrow[\mathbf{r}]{P} C_{\geq h}$*, iff* $U_{\mathbf{r}}(a) \geq b_{h-1}^{\mathbf{r}}$ *for at least one compatible* (U, b)*,*
- $a \in A$ *is necessarily assigned to at most class* C_h*, denoted by* $a \xrightarrow[\mathbf{r}]{N} C_{\leq h}$*, iff* $U_{\mathbf{r}}(a) < b_{h}^{\mathbf{r}}$ *for all compatible* (U, b)*,*

- *$a \in A$ is possibly assigned to at most class C_h, denoted by $a \overset{P}{\underset{\mathbf{r}}{\rightarrow}} C_{\leq h}$, iff $U_{\mathbf{r}}(a) < b^{\mathbf{r}}_h$ for at least one compatible (U, b).*

The above necessary and possible preference relations can be computed as follows:

- $a \overset{N}{\underset{\mathbf{r}}{\rightarrow}} C_{\geq h}$ iff the set of constraints $E^N\left(a \underset{\mathbf{r}}{\rightarrow} C_{\geq h}\right)$ is infeasible or if $\varepsilon\left(a \overset{N}{\underset{\mathbf{r}}{\rightarrow}} C_{\geq h}\right) \leq 0$, where $E^N\left(a \underset{\mathbf{r}}{\rightarrow} C_{\geq h}\right) = E^{A^R} \cup \left\{U_{\mathbf{r}}(a) + \varepsilon \leq b^{\mathbf{r}}_{h-1}\right\}$ and $\varepsilon\left(a \overset{N}{\underset{\mathbf{r}}{\rightarrow}} C_{\geq h}\right) = \max \varepsilon$, s.t. $E^N\left(a \underset{\mathbf{r}}{\rightarrow} C_{\geq h}\right)$;
- $a \overset{P}{\underset{\mathbf{r}}{\rightarrow}} C_{\geq h}$ iff the set of constraints $E^P\left(a \underset{\mathbf{r}}{\rightarrow} C_{\geq h}\right)$ is feasible and $\varepsilon\left(a \overset{P}{\underset{\mathbf{r}}{\rightarrow}} C_{\geq h}\right) > 0$, where $E^P\left(a \underset{\mathbf{r}}{\rightarrow} C_{\geq h}\right) = E^{A^R} \cup \left\{U_{\mathbf{r}}(a) \geq b^{\mathbf{r}}_{h-1}\right\}$ and $\varepsilon\left(a \overset{P}{\underset{\mathbf{r}}{\rightarrow}} C_{\geq h}\right) = \max \varepsilon$, s.t. $E^P\left(a \underset{\mathbf{r}}{\rightarrow} C_{\geq h}\right)$;
- $a \overset{N}{\underset{\mathbf{r}}{\rightarrow}} C_{\leq h}$ iff the set of constraints $E^N\left(a \underset{\mathbf{r}}{\rightarrow} C_{\leq h}\right)$ is infeasible or if $\varepsilon\left(a \overset{N}{\underset{\mathbf{r}}{\rightarrow}} C_{\leq h}\right) \leq 0$, where $E^N\left(a \underset{\mathbf{r}}{\rightarrow} C_{\leq h}\right) = E^{A^R} \cup \{U_{\mathbf{r}}(a) \geq b^{\mathbf{r}}_h\}$ and $\varepsilon\left(a \overset{N}{\underset{\mathbf{r}}{\rightarrow}} C_{\leq h}\right) = \max \varepsilon$, s.t. $E^N\left(a \underset{\mathbf{r}}{\rightarrow} C_{\leq h}\right)$;
- $a \overset{P}{\underset{\mathbf{r}}{\rightarrow}} C_{\leq h}$ iff the set of constraints $E^P\left(a \underset{\mathbf{r}}{\rightarrow} C_{\leq h}\right)$ is feasible and $\varepsilon\left(a \overset{P}{\underset{\mathbf{r}}{\rightarrow}} C_{\leq h}\right) > 0$, where $E^P\left(a \underset{\mathbf{r}}{\rightarrow} C_{\leq h}\right) = E^{A^R} \cup \{U_{\mathbf{r}}(a) + \varepsilon \leq b^{\mathbf{r}}_h\}$ and $\varepsilon\left(a \overset{P}{\underset{\mathbf{r}}{\rightarrow}} C_{\leq h}\right) = \max \varepsilon$, s.t. $E^P\left(a \underset{\mathbf{r}}{\rightarrow} C_{\leq h}\right)$.

Robust hierarchical multiple criteria sorting methods should satisfy some desirable properties. The first two properties are logical properties, that, in fact, have to be satisfied even when there is no hierarchical structure. These properties state that for all $a \in A$ and for any non-elementary criterion $G_{\mathbf{r}}$,

1R) either a is necessarily assigned to at least class C_h, or a is possibly assigned to at most class $C_{h-1}, h \in \{2, \ldots, p\}$,

2R) either a is necessarily assigned to at most class C_k, or a is possibly assigned to at least class $C_{k+1}, k \in \{1, \ldots, p-1\}$.

Conditions **1R)** and **2R)** can be considered as completeness properties for robust hierarchical multiple criteria sorting corresponding to completeness properties for "flat" (non-hierarchical) sorting problems, according to which for all $a \in A$:

1B) either a is assigned to at least class C_h, or a is assigned to at most class $C_{h-1}, h \in \{2, \ldots, p\}$,

2B) either a is assigned to at most class C_k, or a is assigned to at least class $C_{k+1}, k \in \{1, \ldots, p-1\}$.

On the basis of this observation, we shall call properties **1R)** and **2R)**, *first and second completeness properties of robust hierarchical multiple criteria sorting.* Observe that removing the reference to node $G_{\mathbf{r}}$ of the hierarchy tree, the first and the second completeness properties should hold for any robust multiple criteria sorting method.

Other desirable properties are related to the hierarchical nature of the robust sorting, and can be seen as counterparts of the robust multiple criteria sorting of the coherence properties considered in section 2.1.3 for non-hierarchical multiple criteria sorting. The coherence properties for robust hierarchical multiple criteria sorting methods that we shall consider are the following:

- If a is necessarily assigned to at least class C_h in all nodes directly descending from $G_{\mathbf{r}}$, then it is necessarily assigned to at least class C_h in node $G_{\mathbf{r}}$. For example, if student S is necessarily assigned to at least class medium in both Algebra and Analysis, then S has to be assigned to at least class medium also in the node of Mathematics. Let us call this property *first coherence property for robust hierarchical multiple criteria sorting methods*.

- If a is necessarily assigned to at most class C_k in all nodes directly descending from $G_{\mathbf{r}}$, then it is necessarily assigned to at most class C_k in node $G_{\mathbf{r}}$. For example, if student S is necessarily assigned to at most class moderate in both Algebra and Analysis, then S has to be assigned to at most class moderate also in the node of Mathematics. Let us call this property *second coherence property for robust hierarchical multiple criteria sorting methods*.

- If a is necessarily assigned to at least class C_h in all nodes directly descending from $G_{\mathbf{r}}$, with the possible exception of node $\overline{j}$ for which a is possibly assigned to at least class C_h, then a is possibly assigned to at least class C_h in node $G_{\mathbf{r}}$. For example, if student S is assigned to at least class medium necessarily in the node of Algebra, and possibly in the node of Analysis, then S has to be possibly assigned to at least class medium also in the node of Mathematics. Let us call this property *third coherence property for robust hierarchical multiple criteria sorting methods*.

- If a is necessarily assigned to at most class C_k in all nodes directly descending from $G_{\mathbf{r}}$, with the possible exception of node $\overline{j}$ for which a is possibly assigned to at most class C_k, then a is possibly assigned to at most class C_k in node $G_{\mathbf{r}}$. For example, if student S is assigned to at most class moderate necessarily in the node of Algebra, and possibly in the node of Analysis, then S has to be possibly assigned to at most class moderate also in the node of Mathematics. Let us call this property *fourth coherence property for robust hierarchical multiple criteria sorting methods*.

Proposition 2.1.2 given below says that the above two completeness properties, as well as the four coherence properties hold for the hierarchical UTADIS^{GMS} we are proposing. Notice that the four coherence properties are satisfied because statement 1 in Proposition 2.1.1 holds, i.e., because the value thresholds separating the classes in node $G_{\mathbf{r}}$ are equal to the sum of corresponding value thresholds separating the classes in all nodes directly descending from $G_{\mathbf{r}}$.

Proposition 2.1.2. *In any node $G_{\mathbf{r}}$, $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$, of the hierarchy tree,*

1. *For all $a \in A$, and $h = 2, \ldots, p$, either $a \xrightarrow[\mathbf{r}]{N} C_{\geq h}$ or $a \xrightarrow[\mathbf{r}]{P} C_{\leq h-1}$,*
2. *For all $a \in A$, and $k = 1, \ldots, p-1$, either $a \xrightarrow[\mathbf{r}]{N} C_{\leq k}$ or $a \xrightarrow[\mathbf{r}]{P} C_{\geq k+1}$,*
3. *If $a \xrightarrow[(\mathbf{r},j)]{N} C_{\geq h_j}, j = 1, \ldots, n(\mathbf{r})$, then $a \xrightarrow[\mathbf{r}]{N} C_{\geq h}$ where $h = \min\limits_{j=1,\ldots,n(\mathbf{r})} h_j$,*
4. *If $a \xrightarrow[(\mathbf{r},j)]{N} C_{\leq k_j}, j = 1, \ldots, n(\mathbf{r})$, then $a \xrightarrow[\mathbf{r}]{N} C_{\leq k}$ where $k = \max\limits_{j=1,\ldots,n(\mathbf{r})} k_j$,*
5. *If $a \xrightarrow[(\mathbf{r},j)]{N} C_{\geq h_j}, j \in \{1, \ldots, n(\mathbf{r})\} \setminus \{\bar{j}\}$ and $a \xrightarrow[(\mathbf{r},\bar{j})]{P} C_{\geq h_{\bar{j}}}$, then $a \xrightarrow[\mathbf{r}]{P} C_{\geq h}$ where $h = \min\limits_{j=1,\ldots,n(\mathbf{r})} h_j$,*
6. *If $a \xrightarrow[(\mathbf{r},j)]{N} C_{\leq k_j}, j \in \{1, \ldots, n(\mathbf{r})\} \setminus \{\bar{j}\}$, and $a \xrightarrow[(\mathbf{r},\bar{j})]{P} C_{\leq k_{\bar{j}}}$ then $a \xrightarrow[\mathbf{r}]{P} C_{\leq k}$ where $k = \max\limits_{j=1,\ldots,n(\mathbf{r})} k_j$.*

An obvious consequence of Proposition 2.1.2 is that if $a \xrightarrow[(\mathbf{r},j)]{N} C_h, j = 1, \ldots, n(\mathbf{r})$, then $a \xrightarrow[\mathbf{r}]{N} C_h$.

In order to possibly or necessarily assign an alternative $a \in A$ to an interval of classes in node $G_{\mathbf{r}}$ of the hierarchy tree, the following indices can be defined:

$$L_{\mathbf{r}}^{\mathcal{U},P}(a) = \max\left(\{1\} \cup \left\{h \in H : a \xrightarrow[\mathbf{r}]{N} C_{\geq h}\right\}\right), \quad R_{\mathbf{r}}^{\mathcal{U},P}(a) = \min\left(\{p\} \cup \left\{h \in H : a \xrightarrow[\mathbf{r}]{N} C_{\leq h}\right\}\right) \tag{2.4}$$

$$L_{\mathbf{r}}^{\mathcal{U},N}(a) = \max\left(\{1\} \cup \left\{h \in H : a \xrightarrow[\mathbf{r}]{P} C_{\geq h}\right\}\right), \quad R_{\mathbf{r}}^{\mathcal{U},N}(a) = \min\left(\{p\} \cup \left\{h \in H : a \xrightarrow[\mathbf{r}]{P} C_{\leq h}\right\}\right). \tag{2.5}$$

On the basis of Proposition 2.1.2, we can prove the following results:

Proposition 2.1.3. *In any node $G_{\mathbf{r}}$ ($\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$) of the hierarchy tree, and $a \in A$,*

1. $L_{\mathbf{r}}^{\mathcal{U},P}(a) \geq \min\limits_{j=1,\ldots,n(\mathbf{r})} \left\{L_{(\mathbf{r},j)}^{\mathcal{U},P}(a)\right\}$,
2. $R_{\mathbf{r}}^{\mathcal{U},P}(a) \leq \max\limits_{j=1,\ldots,n(\mathbf{r})} \left\{R_{(\mathbf{r},j)}^{\mathcal{U},P}(a)\right\}$.

2.1.5 Application to bank performance rating

Problem context

In order to illustrate the applicability of the proposed approaches, this section presents results from a case study involving bank performance rating, in a context of prudential supervision. Under the existing financial regulatory framework of Basel II, the banking supervisory authorities of each country (e.g. central banks) should conduct performance assessments on a regular basis for banks operating in the country, in order to ensure the stability of the country's banking system. Given that bank defaults are rare events, adequate historical data are usually not available to fit statistical models for estimating the likelihood of financial distress for banking institutions. Therefore, supervisors mainly rely on judgmental peer assessment systems, which take into account all aspects of a bank's operations and risk profile [132, 150, 171]. The application of the MCDA is well-suited in this context, as it provides bank analysts and supervisors with a formal framework and analytic techniques for constructing composite performance indicators, exploring the trade-offs between different risk and performance factors, conducting robustness checks, and exploring stress testing scenarios.

Typically, bank rating systems consider six major dimensions, which define a comprehensive assessment framework referred to as CAMELS:

1) capital adequacy,

2) asset quality,

3) management competence,

4) earning generating ability,

5) liquidity,

6) sensitivity to market risks.

These dimensions are further decomposed into elementary criteria, which are specified according to particular characteristics of the banking system in a country. Thus, the problem has a hierarchical structure and the bank rating assessment process should provide results not only at the comprehensive level, but also at each one of the above main dimensions. The results are commonly expressed in a 5-point rating scale. Thus, the context of bank rating fits well the MCHP sorting framework developed in this study.

Data and criteria

The data used in the analysis are taken from [46] and they originate from the Bank of Greece (the supervisory authority responsible for the Greek banking system). They involve 18 Greek banks between 2001 and 2005 (overall 85 bank-year observations[1], which correspond to the alternatives). The banks have been evaluated on 31 criteria structured in a hierarchical way following the CAMELS framework, as shown in Figure 2.1. The six CAMELS dimensions (Capital-CA, Assets-AS, Management-MC, Earnings-ER, Liquidity-LQ, and Sensitivity to market risks-SM) are the first level criteria, each analyzed through multiple subcriteria in the subsequent level. These subcriteria serve as the elementary decision attributes in the MCHP framework, for which the data are available for the banks in the sample.

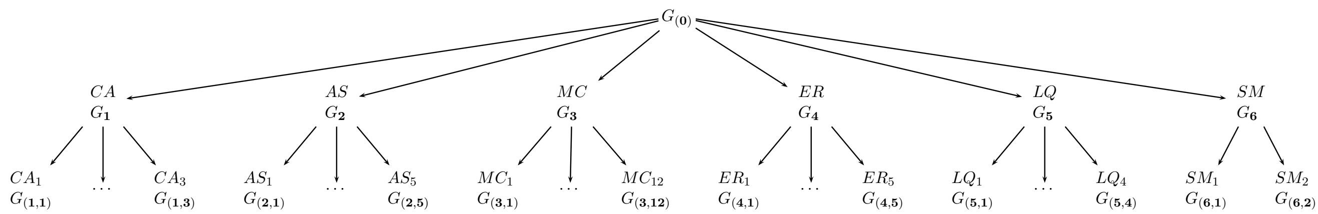


Figure 2.1: Hierarchy of Criteria

The definition of the elementary criteria is given in Table 2.1. These include 17 financial ratios that describe quantitative aspects of bank operation, whereas the remaining 14 criteria describe qualitative issues (but these are still measured on a $0.5-5.5$ cardinal scale defined by analysts at the Bank of Greece, with lower values indicating higher performance). Criteria whose type is indicated in Table 2.1 as "max" are positively related to the performance of banks, whereas minimization criteria are those that are negatively related to bank performance.

For each elementary criterion $g_{\mathbf{t}}$, we considered a linear marginal value function:

$$u_{\mathbf{t}}(a) = u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}})\frac{g_{\mathbf{t}}(a) - x_{\mathbf{t}}^0}{x_{\mathbf{t}}^{m_{\mathbf{t}}} - x_{\mathbf{t}}^0} \tag{2.6}$$

where the best ($x_{\mathbf{t}}^{m_{\mathbf{t}}}$) and worst ($x_{\mathbf{t}}^0$) performances are defined as follows:

$$\begin{aligned}
&\text{Maximization criteria:} \quad x_{\mathbf{t}}^{m_{\mathbf{t}}} = \max\{g_{\mathbf{t}}(a), a \in A\} \quad \text{and} \quad x_{\mathbf{t}}^0 = \min\{g_{\mathbf{t}}(a), a \in A\}\\
&\text{Minimization criteria:} \quad x_{\mathbf{t}}^{m_{\mathbf{t}}} = \min\{g_{\mathbf{t}}(a), a \in A\} \quad \text{and} \quad x_{\mathbf{t}}^0 = \max\{g_{\mathbf{t}}(a), a \in A\}.
\end{aligned}$$

It should be noted that the use of linear marginal value functions in the setting of this case study,

[1] For some banks the data were not available for all years.
The data are available at: `http://www.fel.tuc.gr/BankData.xlsx`

Table 2.1: Evaluation criteria and their indices used in Figure 2.1

Category	Index	Abbr.	Type	Index	Criterion name
Capital	**1**	CA1	Max	$(\mathbf{1}, 1)$	Capital adequacy ratio
		CA2	Min	$(\mathbf{1}, 2)$	TIER II capital / TIER I
		CA3	Min	$(\mathbf{1}, 3)$	Qualitative*
Assets	**2**	AS1	Min	$(\mathbf{2}, 1)$	Risk-weighted assets / Assets
		AS2	Min	$(\mathbf{2}, 2)$	(Non performing loans – Provisions) / Loans
		AS3	Min	$(\mathbf{2}, 3)$	Large exposures / (TIER I + TIER II capital)
		AS4	Min	$(\mathbf{2}, 4)$	[0.5(Non performing loans) – Provisions]/Equity
		AS5	Min	$(\mathbf{2}, 5)$	Qualitative*
Management	**3**	MC1	Min	$(\mathbf{3}, 1)$	Operating expenses / Operating income
		MC2	Min	$(\mathbf{3}, 2)$	Staff cost / Assets
		MC3	Max	$(\mathbf{3}, 3)$	Operating income / Business units
		MC4	Min	$(\mathbf{3}, 4)$	Top management competencies, qualifications and continuity
		MC5	Min	$(\mathbf{3}, 5)$	Managers' experience and competence
		MC6	Min	$(\mathbf{3}, 6)$	Resilience to change, strategy, long term horizon
		MC7	Min	$(\mathbf{3}, 7)$	Management of information systems
		MC8	Min	$(\mathbf{3}, 8)$	Internal control systems
		MC9	Min	$(\mathbf{3}, 9)$	Financial risk management system
		MC10	Min	$(\mathbf{3}, 10)$	Internal processes charter - implementation monitoring
		MC11	Min	$(\mathbf{3}, 11)$	Timely and accurate data collection
		MC12	Min	$(\mathbf{3}, 12)$	Information technology systems
Earnings	**4**	ER1	Max	$(\mathbf{4}, 1)$	Net income / Assets
		ER2	Max	$(\mathbf{4}, 2)$	Net income / Equity
		ER3	Max	$(\mathbf{4}, 3)$	Interest revenue / Assets
		ER4	Max	$(\mathbf{4}, 4)$	Other operating revenue / Assets
		ER5	Min	$(\mathbf{4}, 5)$	Qualitative*
Liquidity	**5**	LQ1	Max	$(\mathbf{5}, 1)$	Cash / Assets
		LQ2	Min	$(\mathbf{5}, 2)$	(Loans – Provisions) / Deposits
		LQ3	Min	$(\mathbf{5}, 3)$	Real funding from credit institutions / Assets
		LQ4	Min	$(\mathbf{5}, 4)$	Qualitative*
Market	**6**	SM1	Min	$(\mathbf{6}, 1)$	Risk-weighted assets II / Risk-weighted Assets (I & II)
		SM2	Min	$(\mathbf{6}, 2)$	Qualitative*

* Undisclosed criteria related to qualitative aspects of the banks' operation

is actually in accordance with the CAMELS modeling framework as implemented by the Bank of Greece. Furthermore, similar linear scoring and risk monitoring systems are widely used by bank supervisory agencies worldwide.

In accordance with the policy followed by analysts at the Bank of Greece during the period under consideration, the following points are taken into consideration:

- The importance of quantitative criteria should be at least equal to 70%. Even though criteria related to qualitative aspects of bank operation are particularly useful for describing important performance and risk factors in the medium-long term, they clearly entail some subjectivity on the way they are modeled and assessed. On the other hand, financial quantitative criteria, despite their shortcomings (e.g., potential manipulation of accounting reporting standards), are hard data widely used in prudential supervision research and practice all over the world. In that regard, this requirement is imposed to ensure that the resulting evaluation does not overweight the qualitative aspects of bank operation over the actual financial results.

Denoting by $\mathcal{I}_{\mathcal{G}_{Qual}}$ the set of indices of qualitative elementary criteria, that is

$$\mathcal{I}_{\mathcal{G}_{Qual}} = \{(\mathbf{1},3),(\mathbf{2},5),(\mathbf{4},5),(\mathbf{5},4),(\mathbf{6},2)\}$$

and by $\mathcal{I}_{\mathcal{G}_{Quan}}$ the set of indices of all quantitative elementary criteria ($\mathcal{I}_{\mathcal{G}_{Qual}} \cup \mathcal{I}_{\mathcal{G}_{Quan}} = \mathcal{I}_{\mathcal{G}}$), the previous piece of preference information can be translated to the following constraint:

$$\sum_{\mathbf{t}\in\mathcal{I}_{\mathcal{G}_{Quan}}} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) \geq 0.7. \tag{2.7}$$

This implies that a bank having the best performance on all quantitative elementary criteria should have a comprehensive value not less than 0.7.

- Capital and assets are the most important dimensions, whereas market risk is the least important one. Capital adequacy and asset quality are critical factors for ensuring the financial soundness of a bank. They are both closely monitored on a regular basis by supervisors, and actions are taken whenever a bank does not have adequate capital (see for example the stress tests conducted by the European Banking Authority) or when its loan portfolio is particularly troublesome. Liquidity is also an important issue, but during the period of the analysis (2001–2005) there were no indications that liquidity risk could be a critical factor in the foreseeable future for Greek banks. Therefore, liquidity is considered to be of lower importance for this analysis, compared to capital adequacy and asset quality. The same applies to earning power and management competence, too. Earning power is an important dimension for the success of banking institutions as it indicates how they perform in multiple areas. Furthermore, a strong stream of earnings constitutes the first line of defense against loan losses. However, the period of the analysis was a time of transition for Greek banks in terms of their profitability, mainly due to the introduction of the Euro and the adoption of the international accounting standards by the largest banks. Due to the challenges that these issues created in assessing the earnings of Greek banking institutions over the period under consideration, its relative importance was set below capital and assets. On the other hand, management competence is mostly related to qualitative aspects of bank operation, which, as explained above, are given lower priority. Finally, the data set only involves commercial banks, whose exposure to market risks is limited. Therefore, the market risk dimension is assumed to be the least important one among the six criteria categories.

Using the notation introduced in section 2.1.2 and indices of criteria shown in Table 2.1, the given three pieces of preference information can be translated to the following sets of constraints

$$\sum_{\mathbf{t}\in E(G_1)} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) \geq \begin{cases} \sum_{\mathbf{t}\in E(G_3)} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) + \varepsilon, \\ \sum_{\mathbf{t}\in E(G_4)} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) + \varepsilon, \\ \sum_{\mathbf{t}\in E(G_5)} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) + \varepsilon, \\ \sum_{\mathbf{t}\in E(G_6)} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) + \varepsilon, \end{cases} \qquad \sum_{\mathbf{t}\in E(G_2)} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) \geq \begin{cases} \sum_{\mathbf{t}\in E(G_3)} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) + \varepsilon, \\ \sum_{\mathbf{t}\in E(G_4)} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) + \varepsilon, \\ \sum_{\mathbf{t}\in E(G_5)} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) + \varepsilon, \\ \sum_{\mathbf{t}\in E(G_6)} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) + \varepsilon, \end{cases} \tag{2.8}$$

$$\sum_{\mathbf{t}\in E(G_6)} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) \leq \begin{cases} \sum_{\mathbf{t}\in E(G_3)} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) - \varepsilon, \\ \sum_{\mathbf{t}\in E(G_4)} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) - \varepsilon, \\ \sum_{\mathbf{t}\in E(G_5)} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) - \varepsilon, \end{cases} \tag{2.9}$$

where the constraints (2.8) say that criteria categories capital and assets are more important than the other four criteria categories, while the constraints (2.9) say that market risk is the least important criteria category. Let us notice that the constraints saying that market risk is less important than capital and assets are missing in (2.9) since these constraints are already present in (2.8).

Discussion of results

In addition to the above preference information, an expert banking analyst (DM) familiar with the Greek banking sector provided global assessments for a small set of banks, as shown in Table 2.2. These are banks for which the DM was familiar with their strengths and weaknesses over the examined period. For example, alternatives A_3, A_4, and A_5 correspond to a leading Greek bank in terms of its market niche and financial strength over a three years period (2003–2005), alternatives A_7, A_8, and A_9 involve a state-owned bank being in transition towards privatization, whereas A_{16}, A_{17}, and A_{18} correspond to a recently privatized bank that faced significant operating challenges moving to a new corporate plan.

Table 2.2: Initial set of the expert's comprehensive judgments

Alternatives	Class
A_{16}, A_{17}, A_{18}	C_1
A_{10}, A_{21}, A_{22}	C_2
A_7, A_8, A_9	C_3
A_1, A_2, A_6	C_4
A_3, A_4, A_5	C_5

Since each bank can be assigned to one of five classes at the level of macro-criteria and at the comprehensive level, then six thresholds have to be specified for each macro-criterion (b_0^s, b_1^s, b_2^s, b_3^s, b_4^s, b_5^s), such that $b_0^s = 0$ and $b_5^s = \sum_{\mathbf{t}\in E(G_s)} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}})$ for all $s \in \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{5},\mathbf{6}\}$. Consequently, following Proposition 2.1.1, the thresholds for criterion G_0 are obtained as the sum of the corresponding thresholds for the six macro-criteria, that is $b_h^0 = \sum_{s=1}^{6} b_h^s$, for all $h = 0,\ldots,5$.

Having defined the thresholds for the six macro-criteria, the preferences shown in Table 2.2 are translated to constraints as explained in section 2.1.3. For example, the assignment at a comprehensive level of bank A_x to class C_h is translated to the constraints

$$\left.\begin{array}{l} U_0(A_x) \geq b_{h-1}^0, \\ U_0(A_x) - b_h^0 \leq -\varepsilon. \end{array}\right\} \tag{2.10}$$

Consequently, the set E^{A^R} containing the constraints translating the preferences of the DM and the technical constraints will be the following:

$$\left.\begin{array}{l} (2.7)-(2.10), \\ u_{\mathbf{t}}(x_{\mathbf{t}}^0) = 0, \text{ for all } \mathbf{t} \in EL, \text{ and } \sum_{\mathbf{t}\in EL} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) = 1 \\ u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}) \geq u_{\mathbf{t}}(x_{\mathbf{t}}^0), \text{ for all } \mathbf{t} \in EL\ , \\ b_h^{\mathbf{s}} \geq b_{h-1}^{\mathbf{s}} + \varepsilon,\ h = 1,\ldots,5, \text{ for all } \mathbf{s} \in \{\mathbf{1},\ldots,\mathbf{6}\}, \\ b_0^{\mathbf{s}} = 0, \text{ and } b_5^{\mathbf{s}} = \sum_{\mathbf{t}\in E(G_{\mathbf{s}})} u_{\mathbf{t}}(x_{\mathbf{t}}^{m_{\mathbf{t}}}), \text{ for all } \mathbf{s} \in \{\mathbf{1},\ldots,\mathbf{6}\}, \\ b_h^{\mathbf{0}} = \sum_{\mathbf{s}\in\{\mathbf{1},\ldots,\mathbf{6}\}} b_h^{\mathbf{s}}, \text{for all } h = 0,\ldots,5. \end{array}\right\}$$

Note that in this case we do not need the monotonicity constraint $u_{\mathbf{t}}(x_{\mathbf{t}}^k) \geq u_{\mathbf{t}}(x_{\mathbf{t}}^{k-1})$, $k = 1, \ldots, m_{\mathbf{t}}$, for any $\mathbf{t} \in EL$ because, as shown in equation (2.6), we are considering a linear marginal value function for each elementary criterion, and this function is defined by the marginal value $u_{\mathbf{t}}(x_{\mathbf{t}})$ and by the worst and the best performances of the banks on each elementary criterion.
Solving the LP problem $\varepsilon^* = \max \varepsilon$, s.t. E^{A^R}, we find that E^{A^R} is feasible and $\varepsilon^* > 0$. This leads to the conclusion that there are multiple different instances of the preference model compatible with the above comprehensive judgments and preferential inputs. Clearly, the choice of a single decision instance from such limited information is likely to lead to conclusions that are not robust. Combining ROR with the modeling framework of the UTADIS method under the hierarchical structuring of the family of criteria, enables the formulation of results taking into account the full set of possible instances.
Applying (2.4), we computed the lowest and the highest possible class assignment for each alternative. Apart from the seven banks shown in Table 2.3, all the others could be possibly assigned to the whole range of classes. Moreover, applying (2.5), we computed the lowest and highest necessary assignment for each alternative. It appears that the set of necessary assignments is empty for all banks, since $L_{\mathbf{0}}^{\mathcal{U},N} > R_{\mathbf{0}}^{\mathcal{U},N}$ for all of them. It is evident that at this stage of the analysis, the obtained results are not conclusive enough.

Table 2.3: Results after the first stage

Alternatives	$\left[L_{\mathbf{0}}^{\mathcal{U},P}, R_{\mathbf{0}}^{\mathcal{U},P}\right]$
A_{23}, A_{54}, A_{61}, A_{68}	$[C_1, C_4]$
A_{36}, A_{80}, A_{81}	$[C_2, C_5]$

In order to get a more clear recommendation, the expert analyst has to provide more detailed preference information. Then, the DM provided partial judgments involving the main CAMELS dimensions, as shown in Table 2.4. These partial judgments are easier for the DM to define, as each main dimension comprises a much smaller set of criteria compared to the 31 criteria required for the comprehensive assignment decisions provided in the previous stage. The calculation of the new recommendation is performed analogously to the first stage.

With the new preference information, the integration of MCHP with the UTADIS^{GMS} method was employed again to get a new set of assignments. Table 2.5 reports the number of non-reference cases (i.e., banks-year observations not included in the assignments provided by the expert analyst), by the type of their assignment result (range of classes) at the comprehensive level and at all lower-

Table 2.4: Information provided by the expert in the second stage

Capital adequacy		Asset quality		Management competence	
Alternative	Assignment	Alternative	Assignment	Alternative	Assignment
A_{67}	$[C_1, C_2]$	A_{60}	$[C_1, C_2]$	A_{17}	C_2
A_{19}	$[C_2, C_3]$	A_{41}	$[C_2, C_3]$	A_{60}	C_3
A_7	C_3	A_{11}	$[C_3, C_4]$	A_1	C_4
A_1	C_4	A_5	$[C_4, C_5]$		
A_4	C_5				
Earning power		Liquidity		Market risks	
A_{19}	C_1	A_{82}	C_1	A_{76}	$[C_1, C_2]$
A_{20}	C_2	A_{28}	C_2	A_{33}	C_3
A_3	C_3	A_{79}	C_3	A_{22}	C_4
A_{26}	C_4	A_{78}	$[C_3, C_4]$	A_{47}	$[C_4, C_5]$
A_{36}	C_5	A_{55}	C_5		

level dimensions. In addition, the table also presents the mean range of the assignments as an indicator of the imprecision that describes the obtained results. The mean range is calculated from the number of classes in the sets of possible assignments, averaged over all non-reference bank-year observations.

Table 2.5: Summary of possible assignments from the second stage of the analysis (non-reference alternatives)

Assignments	Overall	CA	AS	MC	ER	LQ	SM
C_1	–	–	1	–	–	–	–
$[C_1, C_2]$	–	–	1	–	5	2	2
$[C_1, C_3]$	–	–	–	4	10	26	5
$[C_1, C_4]$	11	–	2	22	9	8	11
$[C_1, C_5]$	40	–	56	18	28	25	–
C_2	–	2	–	–	–	–	–
$[C_2, C_3]$	–	13	–	–	–	–	–
$[C_2, C_4]$	–	2	–	7	–	–	–
$[C_2, C_5]$	16	28	17	24	15	19	–
$[C_3, C_5]$	3	34	4	7	9	–	33
$[C_4, C_5]$	–	3	–	–	3	–	30
C_5	–	–	–	–	1	–	–
Mean range	4.5	3.1	4.6	4.0	3.9	3.9	2.7

It is evident that even with the new information, the sorting decisions at the comprehensive level are still characterized by ambiguity, as 40 (out of 70) cases can be assigned in any of the five rating classes. The examination of the partial assignments for each of the six main dimensions provides some insights on the decomposition of the banks' comprehensive performance and the sources of ambiguity in the assignments at the comprehensive level.

In particular, the partial assignments for capital adequacy (CA) and sensitivity to market risks (SM) are more precise compared to the other dimensions. In terms of capital adequacy, all banks are consistently rated in class C_2 or better (throughout the years), with 37 cases being in at least medium condition (i.e., belonging to categories $C_3 - C_5$). This result is concordant with the characteristics of Greek banks during the period of the analysis, as prior to the outbreak of the Greek sovereign debt crisis in 2010, they have been generally well capitalized.

As far as their sensitivity to market risks is concerned, the banks also performed rather well over the period under consideration. In particular, 63 cases are considered as having at least medium performance on this dimension. There are, however, a few cases corresponding to banks that seem to be exposed to market risks (i.e., their assignment includes the high risk class C_1). These are mostly smaller banks, which have indeed developed some risky investment activities and financial products during that period.

Asset quality seems to be the main factor explaining the ambiguity in the assignment at the comprehensive level. In the majority of cases (56 out of 81), the assignments in this dimension span all five rating classes, which indicates that in order to obtain more precise conclusions on the asset quality dimension, further analysis is required using additional input information. The same applies (yet to a smaller extend) to management competence, earnings, and liquidity.

A further examination of the time trends in the range of the assignments over time (Figure 2.2) reveals that the imprecision in the assignments at the comprehensive level has increased over time. This can be interpreted as a warning signal, as it implies that deriving clear conclusions on the overall performance of the banks became more difficult over the years. This trend was primarily driven by the increasing ambiguity in the evaluations with respect to capital adequacy (after 2002), asset quality (mostly in 2001–2002), and management competence. On the other hand, the imprecision in the evaluations with respect to the market risk dimension followed a declining trend, as the introduction of Greece to the Eurozone area in 2002 and the improving conditions in the global financial markets (particularly after 2003) contributed to the minimization of the exposure of Greek banks to external market risks.

The information derived from the imprecise assignments of the UTADIS^{GMS} method can be further enriched and complemented through the construction of the most discriminant additive value model, which is obtained through the solution of the optimization problem: $\max \varepsilon$, subject to E^{A^R}. Table 2.6 presents the number of assignments with the obtained model, both at the comprehensive

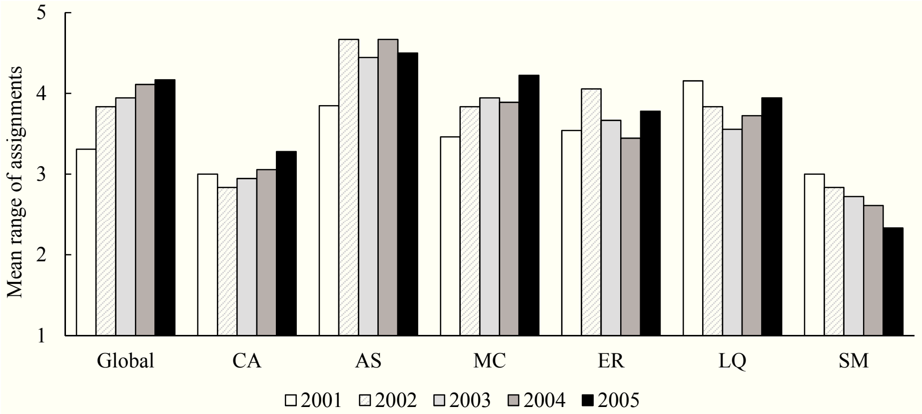


Figure 2.2: Mean ranges of the assignments over time

Table 2.6: Summary of assignment results with the most discriminant value function (number of assignments by each class)

Class	Comprehensively	CA	AS	MC	ER	LQ	SM
C_1	6	0	26	21	8	20	6
C_2	23	22	9	29	24	38	4
C_3	19	17	7	30	29	15	15
C_4	32	43	7	5	15	8	42
C_5	5	3	36	0	9	4	18

level and at the level of the six performance dimensions. According to the results, there are six cases involving very high risk banks (class C_1), five cases of top performing banks (class C_5), whereas most banks are assigned to classes $C_2 - C_5$. The distribution of the assignments for the capital adequacy dimension resembles the assignment at the comprehensive level, whereas in terms of asset quality it is interesting to note that there is a considerable concentration in the two extreme rating classes. This is in accordance with the large number of imprecise assignments in this dimension, as discussed earlier. In terms of management competence and liquidity there is a concentration in classes $C_1 - C_3$ (at most medium performance), whereas the results for market risk verify the remarks made earlier on the low exposure of Greek banks to external market risks as there is a clear concentration in classes $C_4 - C_5$ (above average performance). The Kendall's τ rank correlations between the comprehensive assignment and the partial ones were higher for capital adequacy (0.725) and asset quality (0.653), which is concordant with the information that the expert analyst provided on the high importance of these criteria. The correlations of the comprehensive assignment to those of the other dimensions were lower (0.2−0.3).

Table 2.7 presents further results on the relationship of the imprecise assignments obtained by UTADIS^{GMS} with the ones of the most discriminant model at the comprehensive level. In particular, for banks assigned to different ranges of classes according to UTADIS^{GMS}, we report their mean global values (i.e., performance scores) according to the most representative model (second column), as well as their distribution in the classes resulting from the most discriminant model (frequencies). For instance, the mean performance score for banks assigned to the range of classes $[C_1, C_4]$ is 0.4, and most of such instances (90.9%) are assigned to class C_2 by the most discriminant model. The last row in the table presents the mean comprehensive value for banks assigned to different classes by the most discriminant model. The results indicate that most banks assigned to $[C_1, C_4]$ by UTADIS^{GMS} are considered as low performance banks by the most discriminant model. Banks for which their assignment is completely imprecise according to UTADIS^{GMS} span the whole range of classes with the most discriminant model, but most of them are assigned to the medium performance class C_3. On the other hand, banks assigned to the range of classes $[C_2, C_5]$ and $[C_3, C_5]$ according to UTADIS^{GMS} are assigned to class C_4 by the most discriminant model. However, the mean value of banks in $[C_2, C_5]$ is 0.567, which is very similar to the mean performance (0.559) of banks assigned to C_4 by the most discriminant model (i.e., they resemble typically good banks), whereas banks assigned to $[C_3, C_5]$ have a mean performance value of 0.603, which is higher that the mean of class C_4 but lower that the mean of the top rating class C_5 (0.642).

Table 2.7: The relationship between the results of UTADIS^{GMS} and the assignments of the most discriminant model at the comprehensive level

		Most discriminant assignments				
	Mean value	C_1	C_2	C_3	C_4	C_5
$[C_1, C_4]$	0.400	9.1%	90.9%	–	–	–
$[C_1, C_5]$	0.483	5.0%	25.0%	40.0%	27.5%	2.5%
$[C_2, C_5]$	0.567	–	–	–	93.8%	6.2%
$[C_3, C_5]$	0.603	–	–	–	100.0%	–
	Mean value	0.337	0.423	0.482	0.559	0.642

2.1.6 Conclusions

Several methods are able to deal with multiple criteria sorting problems, but they all assume a single-level organization of the family of criteria. In this paper, we proposed an extension of the MCHP approach to sorting problems with a hierarchical structure of the family of criteria. The MCHP is a methodology that allows the decomposing of decision making problems into smaller dimensions (each taking into account different aspects of the problem). In this context, we introduced modeling formulations that allow the inference of a preference model from decision examples through preference disaggregation techniques based on an additive value function model (UTADIS and UTADIS^{GMS} methods). MCHP combined with UTADIS and UTADIS^{GMS} allows the consideration of both global and partial preference judgments, which adds flexibility to the specification of the input preference information required in the decision aiding process. The applicability of the MCHP-based methods was illustrated through an application regarding the assessment of bank performance.

Future research can be extended towards a number of different directions. First, similar approaches could be considered for other types of preference models for sorting problems, including outranking relation [34], Choquet integral [74], and rule-based models [85, 157]. That would be particularly useful, as it would yield a much more general MCHP framework, covering situations where different aspects of a decision problem require the adoption of different types of models. Group decision making problems can also be considered in such a context. Combinations with simulation methods [106] could also be useful to enhance the assignment recommendations with probabilistic information, whereas further analysis could also focus on building good representative preference models in sorting problems with hierarchical structure, using the techniques presented in previous studies [48, 79]. In addition to these methodological extensions, further testing on other case studies and through experimental computational analyses could provide further insights into the properties of the MCHP-based sorting schemes. Introduction of procedures guiding the elicitation of preference information by the DM in the spirit of active learning would also be useful to reduce the cognitive effort required during the decision aiding process and make such techniques easier to apply in practice.

2.1.7 Appendix

Proof of Proposition 2.1.1

Proof. (1) $\Rightarrow$ (2) Let $a \underset{(\mathbf{r},j)}{\longrightarrow} [C_{h_j}, C_{k_j}]$ for all $j = 1, \ldots, n(\mathbf{r})$. This means that $b^{(\mathbf{r},j)}_{h_j-1} \leq U_{(\mathbf{r},j)}(a) < b^{(\mathbf{r},j)}_{k_j}$ for all $j = 1, \ldots, n(\mathbf{r})$. Let us consider $h = \min_{j=1,\ldots,n(\mathbf{r})} h_j$ and $k = \max_{j=1,\ldots,n(\mathbf{r})} k_j$. For the monotonicity of the thresholds, we shall have for all $j = 1, \ldots, n(\mathbf{r})$ that $b^{(\mathbf{r},j)}_{h-1} \leq b^{(\mathbf{r},j)}_{h_j-1} \leq U_{(\mathbf{r},j)}(a) < b^{(\mathbf{r},j)}_{k_j} \leq b^{(\mathbf{r},j)}_{k}$ for all $j = 1, \ldots, n(\mathbf{r})$ and, consequently, $b^{(\mathbf{r},j)}_{h-1} \leq U_{(\mathbf{r},j)}(a) < b^{(\mathbf{r},j)}_{k}$. Adding up with respect to j, we get

$$b^{\mathbf{r}}_{h-1} = \sum_{j=1}^{n(\mathbf{r})} b^{(\mathbf{r},j)}_{h-1} \leq \sum_{j=1}^{n(\mathbf{r})} U_{(\mathbf{r},j)}(a) < \sum_{j=1}^{n(\mathbf{r})} b^{(\mathbf{r},j)}_{k} = b^{\mathbf{r}}_{k}.$$

From equation (2.1), it follows that $b^{\mathbf{r}}_{h-1} \leq U_{\mathbf{r}}(a) < b^{\mathbf{r}}_{k}$ and, consequently, $a \underset{\mathbf{r}}{\rightarrow} [C_h, C_k]$.

(2) $\Rightarrow$ (3) follows directly by setting $h_j = k_j = h$ for all $j = 1, \ldots, n(\mathbf{r})$.

(3) $\Rightarrow$ (1) follows by contradiction, when we suppose that $b^{\mathbf{r}}_{h} \neq \sum_{j=1}^{n(\mathbf{r})} b^{(\mathbf{r},j)}_{h}$ for some h. This implies that $b^{\mathbf{r}}_{h} > \sum_{j=1}^{n(\mathbf{r})} b^{(\mathbf{r},j)}_{h}$ or $b^{\mathbf{r}}_{h} < \sum_{j=1}^{n(\mathbf{r})} b^{(\mathbf{r},j)}_{h}$.

Let $b^{\mathbf{r}}_{h} > \sum_{j=1}^{n(\mathbf{r})} b^{(\mathbf{r},j)}_{h}$ and $a \in A$ an alternative, such that

$$U_{(\mathbf{r},j)}(a) = b^{(\mathbf{r},j)}_{h} \quad \text{for all} \quad j = 1, \ldots, n(\mathbf{r}). \tag{2.11}$$

Obviously, this implies that $a \underset{(\mathbf{r},j)}{\longrightarrow} C_{h+1}$ for all $j = 1, \ldots, n(\mathbf{r})$. Adding up with respect to j in the two members of equation (2.11), we get $U_{\mathbf{r}}(a) = \sum_{j=1}^{n(\mathbf{r})} U_{(\mathbf{r},j)}(a) = \sum_{j=1}^{n(\mathbf{r})} b^{(\mathbf{r},j)}_{h} < b^{\mathbf{r}}_{h}$ and, consequently, $a \underset{\mathbf{r}}{\rightarrow} C_{\leq h}$, being in contradiction with the hypothesis.

Now, let $b^{\mathbf{r}}_{h} < \sum_{j=1}^{n(\mathbf{r})} b^{(\mathbf{r},j)}_{h}$ and $a \in A$ an alternative, such that

$$U_{(\mathbf{r},j)}(a) = b^{(\mathbf{r},j)}_{h} - \frac{\varepsilon}{n(\mathbf{r})} \quad \text{for all} \quad j = 1, \ldots, n(\mathbf{r}) \tag{2.12}$$

where $\varepsilon > 0$. This choice implies that $a \underset{(\mathbf{r},\mathbf{j})}{\longrightarrow} C_{\leq h}$, for all $j = 1, \ldots, n(\mathbf{r})$. Now, adding up with respect to j in the two members of equation (2.12), we get $U_{\mathbf{r}}(a) = \sum_{j=1}^{n(\mathbf{r})} U_{(\mathbf{r},j)}(a) = \sum_{j=1}^{n(\mathbf{r})} \left[b^{(\mathbf{r},j)}_{h} - \frac{\varepsilon}{n(\mathbf{r})} \right] = \sum_{j=1}^{n(\mathbf{r})} b^{(\mathbf{r},j)}_{h} - \varepsilon$. If we choose ε such that

$$0 < \varepsilon \leq \min \left\{ \min \left\{ n(\mathbf{r}) \cdot \left[b^{(\mathbf{r},j)}_{h} - b^{(\mathbf{r},j)}_{h-1} \right], \ j = 1, \ldots, n(\mathbf{r}) \right\}, \sum_{j=1}^{n(\mathbf{r})} b^{(\mathbf{r},j)}_{h} - b^{\mathbf{r}}_{h} \right\}$$

we obtain that $b^{(\mathbf{r},j)}_{h-1} \leq U_{(\mathbf{r},j)}(a) < b^{(\mathbf{r},j)}_{h}$ for all $j = 1, \ldots, n(\mathbf{r})$[2] and $U_{\mathbf{r}}(a) > b^{\mathbf{r}}_{h}$[3] implying that $a \underset{(\mathbf{r},j)}{\longrightarrow} C_h$ for all $j = 1, \ldots, n(\mathbf{r})$ and $a \underset{\mathbf{r}}{\rightarrow} C_{\geq h+1}$, thus leading to a contradiction. $\square$

Proof of Proposition 2.1.2

[2]Because $\varepsilon \leq \min_{j=1,\ldots,n(\mathbf{r})} n(\mathbf{r}) \left[b^{(\mathbf{r},j)}_{h} - b^{(\mathbf{r},j)}_{h-1} \right]$.

[3]Because $\varepsilon \leq \sum_{j=1}^{n(\mathbf{r})} b^{(\mathbf{r},j)}_{h} - b^{\mathbf{r}}_{h}$ and, consequently $\sum_{j=1}^{n(\mathbf{r})} b^{(\mathbf{r},j)}_{h} - \varepsilon > b^{\mathbf{r}}_{h}$

Proof. 1. Let $a \in A$, $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$ and $h = 2, \ldots, p$ such that not$\left(a \overset{N}{\underset{\mathbf{r}}{\rightarrow}} C_{\geq h}\right)$. This means that there exists at least one (U, b) such that $U_{\mathbf{r}}(a) < b^{\mathbf{r}}_{h-1}$. Therefore $a \overset{P}{\underset{\mathbf{r}}{\rightarrow}} C_{\leq h-1}$. Let us observe that $a \overset{N}{\underset{\mathbf{r}}{\rightarrow}} C_{\geq h}$ and $a \overset{P}{\underset{\mathbf{r}}{\rightarrow}} C_{\leq h-1}$ do not hold simultaneously because, otherwise, a couple $(\overline{U}, \overline{b})$ should exist, such that $\overline{U}_{\mathbf{r}}(a) \geq \overline{b}^{\mathbf{r}}_{h-1}$ and $\overline{U}_{\mathbf{r}}(a) < \overline{b}^{\mathbf{r}}_{h-1}$, which is impossible.

2. Let $a \in A$, $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$ and $k = 1, \ldots, p-1$ such that not$\left(a \overset{N}{\underset{\mathbf{r}}{\rightarrow}} C_{\leq k}\right)$. This means that there exists at least one (U, b) such that $U_{\mathbf{r}}(a) \geq b^{\mathbf{r}}_{k}$. Therefore $a \overset{P}{\underset{\mathbf{r}}{\rightarrow}} C_{\geq k+1}$. Let us observe that $a \overset{N}{\underset{\mathbf{r}}{\rightarrow}} C_{\leq k}$ and $a \overset{P}{\underset{\mathbf{r}}{\rightarrow}} C_{\geq k+1}$ do not hold simultaneously because, otherwise, a couple $(\overline{U}, \overline{b})$ should exist, such that $\overline{U}_{\mathbf{r}}(a) < \overline{b}^{\mathbf{r}}_{k}$ and $\overline{U}_{\mathbf{r}}(a) \geq \overline{b}^{\mathbf{r}}_{k}$, which is impossible.

3. $a \xrightarrow[(\mathbf{r},j)]{N} C_{\geq h_j}$ for all $j = 1, \ldots, n(\mathbf{r})$ implies that $U_{(\mathbf{r},j)}(a) \geq b^{(\mathbf{r},j)}_{h_j-1}$ for all (U, b) and for all $j = 1, \ldots, n(\mathbf{r})$. Considering $h = \min\limits_{j=1,\ldots,n(\mathbf{r})} h_j$, for the monotonicity of the thresholds we have that $U_{(\mathbf{r},j)}(a) \geq b^{(\mathbf{r},j)}_{h-1}$ for all (U, b) and for all j. As a consequence, adding up with respect to j, we get $U_{\mathbf{r}}(a) = \sum\limits_{j=1}^{n(\mathbf{r})} U_{(\mathbf{r},j)}(a) \geq \sum\limits_{j=1}^{n(\mathbf{r})} b^{(\mathbf{r},j)}_{h-1} = b^{\mathbf{r}}_{h-1}$ for all (U, b), which proves point 2.

4. $a \xrightarrow[(\mathbf{r},j)]{N} C_{\leq k_j}$ for all $j = 1, \ldots, n(\mathbf{r})$ implies that $U_{(\mathbf{r},j)}(a) < b^{(\mathbf{r},j)}_{k_j}$ for all (U, b) and for all $j = 1, \ldots, n(\mathbf{r})$. Considering $k = \max\limits_{j=1,\ldots,n(\mathbf{r})} k_j$, for the monotonicity of the thresholds we have that $U_{(\mathbf{r},j)}(a) < b^{(\mathbf{r},j)}_{k}$ for all (U, b) and for all j. As a consequence, adding up with respect to j, we get $U_{\mathbf{r}}(a) = \sum\limits_{j=1}^{n(\mathbf{r})} U_{(\mathbf{r},j)}(a) < \sum\limits_{j=1}^{n(\mathbf{r})} b^{(\mathbf{r},j)}_{k} = b^{\mathbf{r}}_{k}$ for all (U, b), which implies point 3.

5. $a \xrightarrow[(\mathbf{r},j)]{N} C_{\geq h_j}$, for all $j \in \{1, \ldots, n(\mathbf{r})\} \setminus \{\overline{j}\}$ implies that for all (U, b), $U_{(\mathbf{r},j)}(a) \geq b^{(\mathbf{r},j)}_{h_j-1}$ for all $j \in \{1, \ldots, n(\mathbf{r})\} \setminus \{\overline{j}\}$. Analogously, $a \xrightarrow[(\mathbf{r},\overline{j})]{P} C_{\geq h_{\overline{j}}}$ implies that there exists at least one $(\overline{U}, \overline{b})$ such that $\overline{U}_{(\mathbf{r},\overline{j})}(a) \geq \overline{b}^{(\mathbf{r},\overline{j})}_{h_{\overline{j}}-1}$. Considering $h = \min\limits_{j=1,\ldots,n(\mathbf{r})} h_j$, for $(\overline{U}, \overline{b})$ and for the monotonicity of the thresholds we have that $\overline{U}_{(\mathbf{r},j)}(a) \geq \overline{b}^{(\mathbf{r},j)}_{h-1}$ for all $j = 1, \ldots, n(\mathbf{r})$. Adding up with respect to j we get $\overline{U}_{\mathbf{r}}(a) = \sum\limits_{j=1}^{n(\mathbf{r})} \overline{U}_{(\mathbf{r},j)}(a) \geq \sum\limits_{j=1}^{n(\mathbf{r})} \overline{b}^{(\mathbf{r},j)}_{h-1} = \overline{b}^{\mathbf{r}}_{h-1}$, which proves point 4.

6. $a \xrightarrow[(\mathbf{r},j)]{N} C_{\leq k_j}$, for all $j \in \{1, \ldots, n(\mathbf{r})\} \setminus \{\overline{j}\}$ implies that for all (U, b), $U_{(\mathbf{r},j)}(a) < b^{(\mathbf{r},j)}_{k_j}$ for all $j \in \{1, \ldots, n(\mathbf{r})\} \setminus \{\overline{j}\}$. Analogously, $a \xrightarrow[(\mathbf{r},\overline{j})]{P} C_{\leq k_{\overline{j}}}$ implies that there exists at least one $(\overline{U}, \overline{b})$ such that $\overline{U}_{(\mathbf{r},\overline{j})}(a) < \overline{b}^{(\mathbf{r},\overline{j})}_{k_{\overline{j}}}$. Considering $k = \max\limits_{j=1,\ldots,n(\mathbf{r})} k_j$, for $(\overline{U}, \overline{b})$ and for the monotonicity of the thresholds we have that $\overline{U}_{(\mathbf{r},j)}(a) < \overline{b}^{(\mathbf{r},j)}_{k}$ for all $j = 1, \ldots, n(\mathbf{r})$. Adding up with respect to j we get $\overline{U}_{\mathbf{r}}(a) = \sum\limits_{j=1}^{n(\mathbf{r})} \overline{U}_{(\mathbf{r},j)}(a) < \sum\limits_{j=1}^{n(\mathbf{r})} \overline{b}^{(\mathbf{r},j)}_{k} = \overline{b}^{\mathbf{r}}_{k}$, which proves point 5.

□

Proof of Proposition 2.1.3

Proof. 1. Let $L^{\mathcal{U},P}_{(\mathbf{r},j)}(a) = h_j$ for all $j = 1, \ldots, n(\mathbf{r})$. This means that $a \xrightarrow[(\mathbf{r},j)]{N} C_{\geq h_j}$ and not $\left(a \xrightarrow[(\mathbf{r},j)]{N} C_{\geq l} \right)$ with $l > h_j$ for all $j = 1, \ldots, n(\mathbf{r})$. By Proposition 2.1.2 we get $a \xrightarrow[\mathbf{r}]{N} C_{\geq h}$ with $h = \min_{j=1,\ldots,n(\mathbf{r})} h_j$. As a consequence we get the thesis.

2. Let $R^{\mathcal{U},P}_{(\mathbf{r},j)}(a) = k_j$ for all $j = 1, \ldots, n(\mathbf{r})$. This means that $a \xrightarrow[(\mathbf{r},j)]{N} C_{\leq k_j}$ and not $\left(a \xrightarrow[(\mathbf{r},j)]{N} C_{\geq l} \right)$ with $l > k_j$ for all $j = 1, \ldots, n(\mathbf{r})$. By Proposition 2.1.2 we get $a \xrightarrow[\mathbf{r}]{N} C_{\geq k}$ with $k = \max_{j=1,\ldots,n(\mathbf{r})} k_j$. As a consequence we get the thesis.

□

2.2 Robust Ordinal Regression and Stochastic Multiobjective Acceptability Analysis in Multiple Criteria Hierarchy Process for the Choquet integral preference model

2.2.1 Introduction

Multiple Criteria Decision Aiding (MCDA) helps Decision Makers in solving choice, ranking and sorting problems concerning a set of alternatives evaluated on multiple criteria (see [51] for a collection of state-of- the-art surveys on MCDA). Taking into account preferences of a particular Decision Maker (DM), in choice problems, a subset of best alternatives has to be chosen; in ranking problems, alternatives have to be partially or totally rank ordered from the best to the worst, while in sorting problems each alternative has to be assigned to one or more contiguous preferentially ordered classes. In order to deal with any of these problems, the evaluations of the alternatives on the considered criteria have to be aggregated by a preference model, which can be either a value function [109], or an outranking relation [27, 55], or a set of decision rules [82, 156].

Nowadays, MCDA is facing three important methodological challenges: handling a complex structure of criteria, dealing with interactions bewteen criteria, and reducing the cognitive effort of the DMs in interaction with MCDA methods. These challenges are usually handled separately, however, they often concern the same decision problem.

In particular, with respect to the complex structure of criteria having the form of a hierarchy, the Analytic Hierarchy Process (AHP) [146], and then the Multiple Criteria Hierarchy Process (MCHP) [37] have been proposed. While AHP requires preference information at all levels of the hierarchy in the form of exhaustive pairwise comparisons, and provides recommendations at the comprehensive level only, MCHP accepts a partial preference information in form of pairwise comparisons of some alternatives at some levels of the hierarchy, and provides recommendations at all levels.

As to the challenge of interaction, it is present when evaluation criteria are not mutually preferentially independent [109]. To deal with interactions, MCDA methods use non additive integrals, such as the Choquet integral (see [31] for the Choquet integral definition, and [68] for the application of non additive integrals in MCDA), the Sugeno integral [162], and some of their generalizations [72, 80, 89, 121]). The preferential independence condition has also been smoothed in multiplicative and multilinear utility functions [109], but due to the high number of parameters that have to be elicited from the DM, their use has not been very successful in real world applications [161].

Moreover, the interaction between criteria has been recently considered in the ELECTRE methods [52] and in PROMETHEE methods [34]. It was also handled in artificial intelligence approaches, by weakening the preference independence condition in GAI-networks [67], as well as UCP-networks [17]. They are based on the concept of Generalized Additive Independence (GAI) decomposition introduced by Fishburn [56], which permits to aggregate performances on considered criteria through the sum of marginal utilities related to subsets of criteria. Yet another approach, recently proposed to deal with the interaction between criteria [88] is based on an enriched additive value function that is decomposed of the usual sum of marginal value functions related to each one of considered criteria and some additional terms expressing a bonus (in case of positive interaction) or a penalty (in case of negative interaction), incurred for interaction between some criteria. In this approach, the pairs of criteria for which there exists a positive or negative interaction are inferred through ordinal regression on the basis of preference information given by the DM on some reference alternatives.

The aforementioned aspects of hierarchy and interaction of criteria have been jointly analyzed and described in the hierarchical Choquet integral preference model [5]. Other studies devoted to modeling the hierarchy of criteria within the Choquet integral preference model can be found in [60, 61, 62, 63, 129, 130, 163]. Let us remark that their multi-step Choquet integral is different from our approach, since it requires the definition of a capacity at each node of the hierarchy of criteria. Consequently, their method considers Choquet integrals resulting from aggregation of Choquet integrals at the subsequent level of the hierarchy, which is not the case of our approach.

As to the challenge of reducing the cognitive effort of the DM, one can observe the trend of abandoning direct elicitation of preference model parameters in favor of an indirect elicitation of preferences. In the direct elicitation, the DM is expected to provide values of all parameters of the considered preference model, while in the indirect elicitation, the DM is expected to provide preference information in the form of pairwise comparisons between some alternatives or criteria. There are known two MCDA methodologies based on the indirect elicitation of preferences, which explore the whole set of preference model parameters compatible with the preference information provided by the DM. These are the Robust Ordinal Regression (ROR) (see [86] for the paper introducing ROR, and [35, 36] for surveys) and the Stochastic Multiobjective Acceptability Analysis (SMAA) (see [113] for the paper introducing SMAA, and [164] for a survey).

In this paper, we undertake all these three challenges together, combining the use of MCHP with the Choquet integral preference model on one hand and application of ROR and SMAA on the other hand. This combination is not straightforward, however, because it does not consist in chaining these three methods as they are, but in joint application of all of them, which needs some non-trivial adaptations. In this way, we extend the study presented in [5] by considering two new aspects:

- application of ROR to identify all instances of the Choquet integral preference model being compatible with the preference information provided by the DM; due to hierarchical structure of criteria, the DM can express preference information at a particular level of the hierarchical decomposition of the problem; in exchange, ROR provides robust recommendation in terms of necessary and possible preference relations at all levels of the hierarchy of criteria;
- application of SMAA to compute the frequency with which an alternative gets a particular position in the recommended ranking or the frequency with which an alternative is preferred to another one, at all levels of the hierarchy of criteria.

Let us observe that the methodology presented in this paper is not just a simple sum of the aforementioned three approaches, because MCHP requires that the Choquet integral preference model, SMAA and ROR are applied in all nodes of the hierarchy of criteria in a different way than in case of a flat structure of criteria; the hierarchy requires a coordination of calculations in particular nodes, and moreover, the preference information does not need to be given in all nodes. Moreover, the approach is really adaptive with respect to the complexity of the decision problem considered, since on one hand, it permits decomposition of complex problems due to hierarchical structure of criteria and, on the other hand, it permits to adapt the Choquet integral from 1-additive form (linear) to k-additive form, depending on the preference information provided by the DM. Another aspect that we would like to underline here and that will be clear in the next sections is that the extension of the MCHP to the Choquet integral preference model does not require more parameters than the application of the Choquet integral preference model in case of a flat structure of criteria. Indeed, the application of the Choquet integral in case of criteria structured in a hierarchical way requires only the definition of a capacity on the set of elementary subcriteria and not of a capacity on each node of the hierarchy. Indeed, the capacities on the different nodes of the hierarchy can be easily obtained by the capacity defined on the elementary subcriteria only.
The highlights characterizing the approach presented in this paper, are summarized briefly in the following paragraphs.

At the input, the DM is asked to provide the following preference information:

- comparisons related to importance and interaction of macro-criteria as well as between some elementary criteria, not necessarily belonging to the same macro-criterion;

- preference comparisons between alternatives at a comprehensive level as well as considering only a macro-criterion and, therefore, a particular aspect of the problem at hand.

At the output, the DM gets, we get the following results again with respect to each node of the hierarchy as well as at a comprehensive level:

- necessary and possible preference relations resulting from NAROR;
- all the probabilistic indices supplied by SMAA applied to the k-additive Choquet integral preference model;
- the rankings of the alternatives, by applying the Choquet integral preference model assuming the barycenter of the capacities compatible with the preference information provided by the DM.

The paper is organized as follows. In Section 2, we introduce some basic concepts relative to the Choquet integral preference model, MCHP, hierarchical Choquet integral preference model, ROR and SMAA. In Section 3, the proposed methodology, combining SMAA and ROR applied to the hierarchical Choquet integral preference model, is presented. A real world multicriteria problem, related to the ranking of universities, illustrates the considered methodology in Section 4. Conclusions are drawn and some future directions of research are provided in Section 5.

2.2.2 Basic concepts

In this section, we introduce some basic concepts used further in the paper. In subsection 3.1.2, we present the Choquet integral preference model. In subsections 2.2.2 and 2.2.2, we recall ROR applied to the Choquet integral (called NAROR), and SMAA, respectively, while in subsection 2.2.2, a description of the hierarchical Choquet integral preference model is presented together with an example (subsection 2.2.2).

The Choquet integral, preference model

Let $G = \{g_1, \ldots, g_n\}$ be the set of evaluation criteria and 2^G the set of all subsets of G; a capacity on 2^G is a function $\mu : 2^G \rightarrow [0, 1]$ such that $\mu(\emptyset) = 0$, $\mu(G) = 1$ (normalization constraints) and $\mu(T) \leq \mu(R)$ for all $T \subseteq R \subseteq G$ (monotonicity constraints). The Möbius representation of the capacity μ is the function $m : 2^G \rightarrow \mathbb{R}$, such that, for all $R \subseteq G$,

$$\mu(R)=\sum_{T\subseteq R} m(T). \tag{2.13}$$

Let also A be a set of alternatives. Given an alternative $a\in A$ and a capacity μ, the Choquet integral of a is defined as

$$C_\mu(a)=\sum_{i=1}^{n}\left[g_{(i)}(a)-g_{(i-1)}\left(a\right)\right]\mu\left(N_i\right),$$

where $_{(\cdot)}$ stands for a permutation of the indices of criteria, such that $0=g_{(0)}(a)=g_{(1)}\left(a\right)\leq\ldots\leq g_{(n)}\left(a\right)$, with $N_i=\{(i),....,(n)\}$, $i=1,..,n$. In the following, we suppose that all criteria are of the gain type.

Using the Möbius representation of μ, and without reordering the criteria, the Choquet integral of a is therefore redefined as

$$C_\mu(a)=\sum_{T\subseteq G} m(T)\min_{i\in T} g_i\left(a\right).$$

Since in case of interacting criteria the importance of the criterion i, as well as its interaction with other criteria, do not depend on its importance as singleton only, but also on its contribution to all coalitions of criteria, we recall the *Shapley value* [154]

$$\varphi\left(\{i\}\right)=\sum_{T\subseteq G:\, i\notin T}\frac{(|G-T|-1)!|T|!}{|G|!}\left[\mu(T\cup\{i\})-\mu(T)\right], \tag{2.14}$$

and the *interaction index* [128]

$$\varphi\left(\{i,j\}\right)=\sum_{T\subseteq G:\, i,j\notin T}\frac{(|G-T|-2)!|T|!}{(|G|-1)!}\left[\mu(T\cup\{i,j\})-\mu(T\cup\{i\})-\mu(T\cup\{j\})+\mu(T)\right]. \tag{2.15}$$

Using the Möbius representation of capacity μ, equations (2.14) and (2.15) can be formulated as follows [76]

$$\varphi(\{i\})=\sum_{A\subseteq G:\, i\in A}\frac{m(A)}{|A|} \tag{2.16}$$

and

$$\varphi(\{i,j\})=\sum_{\{i,j\}\subseteq A\subseteq G}\frac{m(A)}{|A|-1}. \tag{2.17}$$

A direct application of the Choquet integral preference model implies the elicitation of $2^{|G|}-2$

parameters $\mu(T)$ (one for each subset $T \subseteq G$, apart from $T = \emptyset$ and $T = G$, since $\mu(\emptyset) = 0$ and $\mu(G) = 1$). As the inference of all these parameters is cognitively hard, the concept of *q-additive* capacity has been defined in [69]. A capacity is q-additive if $m(T) = 0$ for all $T \subseteq G$, such that $|T| > q$. In real world applications, it is enough considering 2-additive capacities only. The use of a 2-additive capacity involves knowledge of $n + \binom{n}{2}$ parameters only: a value $m(\{i\})$ for each criterion i and a value $m(\{i,j\})$ for each couple of criteria $\{i,j\}$. Considering the Möbius representation m of a 2-additive capacity μ, normalization and monotonicity constraints have the following form

1c) $m(\emptyset) = 0, \ \sum_{i \in G} m(\{i\}) + \sum_{\{i,j\} \subseteq G} m(\{i,j\}) = 1,$

2c) $\begin{cases} m(\{i\}) \geq 0, \ \forall i \in G, \\ m(\{i\}) + \sum_{j \in T} m(\{i,j\}) \geq 0, \ \forall i \in G \text{ and } \forall \, T \subseteq G \setminus \{i\}, \, T \neq \emptyset, \end{cases}$

while the Choquet integral of $a \in A$ can be computed as

$$C_\mu(a) \quad = \quad \sum_{i \in G} m(\{i\}) \, g_i(a) + \sum_{\{i,j\} \subseteq G} m(\{i,j\}) \min\{g_i(a), g_j(a)\}. \tag{2.18}$$

Equations (2.16) and (2.17) expressing the Shapley value and the interaction index can be, therefore, rewritten in the following way:

$$\varphi(\{i\}) = m(\{i\}) + \sum_{j \in G \setminus \{i\}} \frac{m(\{i,j\})}{2}, \tag{2.19}$$

$$\varphi(\{i,j\}) = m(\{i,j\}). \tag{2.20}$$

Non Additive Robust Ordinal Regression (NAROR)

NAROR [8] belongs to the family of ROR methods (see [35, 36, 86]). In NAROR, the DM is asked to give the following type of preference information on a subset $A^* \subseteq A$ of reference alternatives (s)he knows well:

- a is preferred to b, denoted by $a \succ b$ (translated to the constraint $C_\mu(a) \geq C_\mu(b) + \varepsilon$);
- a is indifferent to b, denoted by $a \sim b$ ($C_\mu(a) = C_\mu(b)$);

- a is preferred to b more than c is preferred to d, denoted by $(a,b) \succ^* (c,d)$ ($C_\mu(a) - C_\mu(b) \geq C_\mu(c) - C_\mu(d) + \varepsilon$ and $C_\mu(c) - C_\mu(d) \geq \varepsilon$);

- the intensity of preference between a and b is the same of the intensity of preference between c and d, denoted by $(a,b) \sim^* (c,d)$ ($C_\mu(a) - C_\mu(b) = C_\mu(c) - C_\mu(d)$),

where $a, b, c, d \in A^*$.

Moreover, differently from other ROR methods, in NAROR the DM can provide also some preference information on criteria $i, j, l, k \in G$, such as:

- criterion i is more important than criterion j, denoted by $g_i \succ g_j$ (translated to the constraint $\varphi(\{i\}) \geq \varphi(\{j\}) + \varepsilon$);

- criteria i and j are indifferent, denoted by $g_i \sim g_j$ ($\varphi(\{i\}) = \varphi(\{j\})$);

- criteria i and j are positively (negatively) interacting ($\varphi(\{i,j\}) \geq \varepsilon$ $(\leq -\varepsilon)$);

- i is preferred to j more than l is preferred to k, denoted by $(g_i, g_j) \succ^* (g_l, g_k)$ ($\varphi(\{i\}) - \varphi(\{j\}) \geq \varphi(\{l\}) - \varphi(\{k\}) + \varepsilon$ and $\varphi(\{l\}) - \varphi(\{k\}) \geq \varepsilon$);

- the difference of importance between i and j is the same as the difference of importance between l and k, denoted by$(g_i, g_j) \sim^* (g_l, g_k)$ ($\varphi(\{i\}) - \varphi(\{j\}) = \varphi(\{l\}) - \varphi(\{k\})$).

In the above constraints, ε is an auxiliary variable used to convert the strict inequalities into weak inequalities; for example $C_\mu(a) \geq C_\mu(b) + \varepsilon$ is the translation of $C_\mu(a) > C_\mu(b)$.

At the output of NAROR, two preference relations, one *necessary* $\succsim^N$ and another *possible* $\succsim^P$, are presented to the DM:

$$a \succsim^N b \text{ iff } C_\mu(a) \geq C_\mu(b) \text{ for all } \textit{compatible capacities},$$

$$a \succsim^P b \text{ iff } C_\mu(a) \geq C_\mu(b) \text{ for at least one } \textit{compatible capacity},$$

where a *compatible capacity* is a set of Möbius measures for which the preference information provided by the DM is restored.

Denoting by E^{DM} the set of above constraints translating the DM's preference information together with the monotonicity and normalization constraints **1c)** and **2c)**, the existence of a *compatible capacity* is checked by solving the following linear programming problem:

$$\varepsilon^* = \max \varepsilon, \text{ subject to } E^{DM}.$$

If E^{DM} is feasible and $\varepsilon^* > 0$, then there exists at least one compatible capacity, otherwise there exists some inconsistency in the preferences provided by the DM that could be identified by using one of the methods presented in [125].

The two following sets of constraints,

$$\left.\begin{array}{l} C_\mu(b) \geq C_\mu(a) + \varepsilon, \\ E^{DM} \end{array}\right\} E^N(a,b), \qquad \left.\begin{array}{l} C_\mu(a) \geq C_\mu(b) \\ E^{DM} \end{array}\right\} E^P(a,b)$$

are used to compute the necessary and the possible preference relation between alternatives a and b, $a, b \in A$. In particular, the necessary preference relation holds between a and b if $E^N(a,b)$ is infeasible or $\varepsilon^N \leq 0$, where $\varepsilon^N = \max \varepsilon$, subject to $E^N(a,b)$. Analogously, the possible preference relation holds between a and b if $E^P(a,b)$ is feasible and $\varepsilon^P > 0$, where $\varepsilon^P = \max \varepsilon$, subject to $E^P(a,b)$.

Stochastic Multiobjective Acceptability Analysis (SMAA)

SMAA [113, 115] is a family of MCDA methods which take into account uncertainty or imprecision on the evaluations and preference model parameters. In this section we describe SMAA-2 [115], since our proposed methodology also regards ranking problems.
The most common value function used in SMAA-2 is the linear one

$$U(a_k, w) = \sum_{i=1}^{n} w_i g_i(a_k)$$

where $w \in W = \{(w_1, \ldots, w_n) \in \mathbb{R}^n \colon w_i \geq 0 \text{ and } \sum_{i=1}^{n} w_i = 1\}$. In SMAA methods, the indirect preference information is composed of two probability distributions, f_χ and f_W, defined on the evaluation space χ and on the weight space W, respectively.
Defining the rank function

$$rank(k, \xi, w) = 1 + \sum_{h \neq k} \rho\left(U(\xi_h, w) > U(\xi_k, w)\right),$$

(where $\rho(false) = 0$ and $\rho(true) = 1$) that, for all $a_k \in A$, $\xi \in \chi$ and $w \in W$ gives the rank position of alternative a_k, SMAA-2 computes the set of weights of criteria for which alternative a_k assumes rank $r = 1, 2, \ldots, n$, as follows:

$$W_k^r(\xi) = \{w \in W : rank(k, \xi, w) = r\}.$$

The following further indices are computed in SMAA-2:

- *The rank acceptability index* that measures the variety of different parameters compatible with the DM's preference information giving to the alternative a_k the rank r:

$$b_k^r = \int_{\xi \in \chi} f_\chi(\xi) \int_{w \in W_k^r(\xi)} f_W(w)\ dw\ d\xi;$$

  b_k^r gives the probability that alternative a_k has rank r, and it is within the range $[0, 1]$.

- *The central weight vector* that describes the preferences of a typical DM giving to a_k the best position:

$$w_k^c = \frac{1}{b_k^1} \int_{\xi \in \chi} f_\chi(\xi) \int_{w \in W^1(\xi)} f_W(w) w\ dw\ d\xi;$$

- *The pairwise winning index* that is defined as the frequency that an alternative a_h is preferred to an alternative a_k in the space of weight vectors:

$$p_{hk} = \int_{w \in W} f_W(w) \int_{\xi \in \chi : u(\xi_h, w) > u(\xi_k, w)} f_\chi(\xi) d\xi\ dw.$$

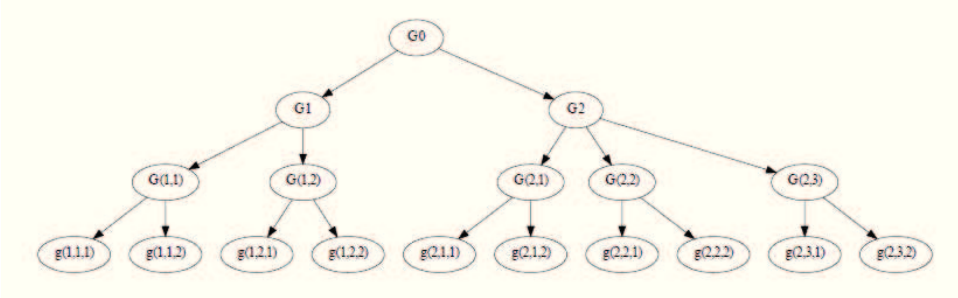
From a computational point of view, the multidimensional integrals defining the considered indices are estimated by using the Monte Carlo method. Let us note that, recently, the potentialities of SMAA and the Choquet integral preference model have been combined in [3] and further investigated in [4].

Multiple Criteria Hierarchy Process (MCHP) and the Choquet integral preference model

In MCHP, the evaluation criteria are not all considered at the same level but they are structured in a hierarchical way. This means that one considers a root criterion (the comprehensive objective) and a set of subcriteria branching successively, as shown in Figure 2.3.
The following notation will be used in the paper:

- $\mathcal{G}$ denotes the set of all criteria at all considered levels, while $\mathcal{I}_\mathcal{G}$ is the set of indices of all criteria in the hierarchy;

Figure 2.3: Example of a hierarchy of criteria. G_0 is the root criterion, G_1 and G_2 are the first level subcriteria while 10 elementary subcriteria are in the last level of the hierarchy.



- EL is the set of indices of the elementary subcriteria, that is criteria located at the last level of the hierarchy and on which the alternatives are evaluated (in Figure 2.3, $EL = \{(1,1,1),(1,1,2),(1,2,1),(1,2,2),(2,1,1),(2,1,2),(2,2,1),(2,2,2),(2,3,1),(2,3,2)\}$);

- $G_{\mathbf{r}}$ is a generic criterion in the hierarchy, while $G_{(\mathbf{r},1)},\ldots,G_{(\mathbf{r},n(\mathbf{r}))}$ are the subcriteria of criterion $G_{\mathbf{r}}$ in the subsequent level (in Figure 2.3, $G_{(\mathbf{2},1)}$, $G_{(\mathbf{2},2)}$ and $G_{(\mathbf{2},3)}$ are the subcriteria of $G_{\mathbf{2}}$ in the subsequent level);

- $E(G_{\mathbf{r}})$ is the set of indices of elementary subcriteria descending from $G_{\mathbf{r}}$ (in Figure 2.3, $E(G_{\mathbf{1}}) = \{(\mathbf{1},1,1),(\mathbf{1},1,2),(\mathbf{1},2,1),(\mathbf{1},2,2)\}$);

- Given $\mathcal{F} \subseteq \mathcal{G}$, $E(\mathcal{F}) = \cup_{G_{\mathbf{r}}\in\mathcal{F}} E(G_{\mathbf{r}})$ is the set of all elementary subcriteria descending from at least one criterion in $\mathcal{F}$ (in Figure 2.3, considering $\mathcal{F} = \left\{G_{(\mathbf{1},1)}, G_{(\mathbf{2},3)}\right\}$, then $E(\mathcal{F}) = \{(\mathbf{1},1,1),(\mathbf{1},1,2),(\mathbf{2},3,1),(\mathbf{2},3,2)\}$);

- $\mathcal{G}_{\mathbf{r}}^k$ is the set of subcriteria of $G_{\mathbf{r}}$ located at level k (in Figure 2.3, $\mathcal{G}_{\mathbf{1}}^2 = \left\{G_{(\mathbf{1},1)}, G_{(\mathbf{1},2)}\right\}$, while $\mathcal{G}_{\mathbf{1}}^3 = \left\{g_{(\mathbf{1},1,1)}, g_{(\mathbf{1},1,2)}, g_{(\mathbf{1},2,1)}, g_{(\mathbf{1},2,2)}\right\}$).

Given a capacity μ defined on the power set of EL, a criterion $G_{\mathbf{r}}$ with $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \cap \mathbb{N}^h$ (that is, $G_{\mathbf{r}}$ is a criterion located at level h of the hierarchy) and $k = h+1,\ldots,l$, where l is the number of levels in the hierarchy tree (for example, $l = 3$ in Figure 2.3), we can define a capacity on the power set of $\mathcal{G}_{\mathbf{r}}^k$

$$\mu_{\mathbf{r}}^k : 2^{\mathcal{G}_{\mathbf{r}}^k} \to [0,1] \tag{2.21}$$

such that

$$\mu_{\mathbf{r}}^k(\mathcal{F}) = \frac{\mu\left(E\left(\mathcal{F}\right)\right)}{\mu\left(E\left(G_{\mathbf{r}}\right)\right)} \tag{2.22}$$

for all $\mathcal{F} \subseteq \mathcal{G}_{\mathbf{r}}^{k}$.

According to equation (2.22), the capacity $\mu_{\mathbf{r}}^{k}$ can be written in terms of the capacity μ defined on the power set of EL.

Considering criterion $G_{\mathbf{r}}$, $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus EL$ at any but the last level of the hierarchy, and the capacity μ defined on the power set of EL, the Choquet integral of alternative $a \in A$ on criterion $G_{\mathbf{r}}$ can be computed as

$$C_{\mu_{\mathbf{r}}}(a) = \frac{C_{\mu}(a_{\mathbf{r}})}{\mu(E(G_{\mathbf{r}}))} \tag{2.23}$$

where $a_{\mathbf{r}}$ is a fictitious alternative having the same evaluations as a on elementary subcriteria from $E(G_{\mathbf{r}})$ and null evaluation on elementary subcriteria from outside $E(G_{\mathbf{r}})$, i.e., $g_{\mathbf{t}}(a_{\mathbf{r}}) = g_{\mathbf{t}}(a)$ if $\mathbf{t} \in E(G_{\mathbf{r}})$ and $g_{\mathbf{t}}(a_{\mathbf{r}}) = 0$ if $\mathbf{t} \notin E(G_{\mathbf{r}})$.

Starting from equations (2.14) and (2.15), and considering a criterion $G_{\mathbf{r}}$, $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus \{EL\}$, we can define the Shapley value of criterion $G_{(\mathbf{r},w)}$ and the interaction index between criteria $G_{(\mathbf{r},w_1)}$ and $G_{(\mathbf{r},w_2)}$, with $G_{(\mathbf{r},w)}$, $G_{(\mathbf{r},w_1)}$, $G_{(\mathbf{r},w_2)} \in \mathcal{G}_{\mathbf{r}}^{k}$, as follows:

$$\varphi_{\mathbf{r}}^{k}\left(\left\{G_{(\mathbf{r},w)}\right\}\right) = \sum_{T \subseteq \mathcal{G}_{\mathbf{r}}^{k} \setminus \left\{G_{(\mathbf{r},w)}\right\}} \frac{\left(|\mathcal{G}_{\mathbf{r}}^{k} \setminus T| - 1\right)!|T|!}{|\mathcal{G}_{\mathbf{r}}^{k}|!} \left[\mu_{\mathbf{r}}^{k}\left(T \cup \left\{G_{(\mathbf{r},w)}\right\}\right) - \mu_{\mathbf{r}}^{k}(T)\right], \tag{2.24}$$

$$\varphi_{\mathbf{r}}^{k}\left(\left\{G_{(\mathbf{r},w_1)}, G_{(\mathbf{r},w_2)}\right\}\right) = \sum_{T \subseteq \mathcal{G}_{\mathbf{r}}^{k} \setminus \left\{G_{(\mathbf{r},w_1)},\, G_{(\mathbf{r},w_2)}\right\}} \frac{\left(|\mathcal{G}_{\mathbf{r}}^{k} \setminus T| - 2\right)!|T|!}{\left(|\mathcal{G}_{\mathbf{r}}^{k}| - 1\right)!}. \tag{2.25}$$

As for the capacity μ defined on the power set of G, we can define analogously the Möbius representation $m_{\mathbf{r}}^{k} : 2^{\mathcal{G}_{\mathbf{r}}^{k}} \rightarrow [0,1]$ of the capacity $\mu_{\mathbf{r}}^{k}$, such that

$$\mu_{\mathbf{r}}^{k}(\mathcal{F}) = \sum_{T \subseteq \mathcal{F}} m_{\mathbf{r}}^{k}(T) \tag{2.26}$$

for all $\mathcal{F} \subseteq \mathcal{G}_{\mathbf{r}}^{k}$.

By considering the Möbius representation $m_{\mathbf{r}}^{k}$ of the capacity $\mu_{\mathbf{r}}^{k}$, equations (2.24) and (2.25) can be rewritten as follows:

$$\varphi_{\mathbf{r}}^{k}\left(\left\{G_{(\mathbf{r},w)}\right\}\right) = \sum_{\mathcal{F} \subseteq \mathcal{G}_{\mathbf{r}}^{k}:\, G_{(\mathbf{r},w)} \in \mathcal{F}} \frac{m_{\mathbf{r}}^{k}(\mathcal{F})}{|\mathcal{F}|} \tag{2.27}$$

$$\varphi_{\mathbf{r}}^{k}\left(\left\{G_{(\mathbf{r},w_1)}, G_{(\mathbf{r},w_2)}\right\}\right) = \sum_{\mathcal{F} \subseteq \mathcal{G}_{\mathbf{r}}^{k}:\, G_{(\mathbf{r},w_1)},\, G_{(\mathbf{r},w_2)} \in \mathcal{F}} \frac{m_{\mathbf{r}}^{k}(\mathcal{F})}{|\mathcal{F}| - 1} \tag{2.28}$$

where $G_{(\mathbf{r},w)}, G_{(\mathbf{r},w_1)}, G_{(\mathbf{r},w_2)} \in \mathcal{G}_{\mathbf{r}}^k$.

The Möbius transformation $m_{\mathbf{r}}^k$ of the capacity $\mu_{\mathbf{r}}^k$ can be written in terms of the Möbius transformation m of the capacity μ, as stated in the following proposition.

Proposition 2.2.1. *Let μ be, a capacity defined on 2^{EL}, and m its Möbius representation. Let $G_{\mathbf{r}} \in \mathcal{G}$, $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus \{EL\}$ with $\mu_{\mathbf{r}}^k$ being a capacity defined on $2^{\mathcal{G}_{\mathbf{r}}^k}$ and $m_{\mathbf{r}}^k$ its Möbius representation; then for all $\mathcal{F} = \left\{G_{(\mathbf{r},w_1)}, \ldots, G_{(\mathbf{r},w_\alpha)}\right\} \subseteq \mathcal{G}_{\mathbf{r}}^k$,*

$$m_{\mathbf{r}}^k(\mathcal{F}) = m_{\mathbf{r}}^k\left(\left\{G_{(\mathbf{r},w_1)}, \ldots, G_{(\mathbf{r},w_\alpha)}\right\}\right) = \frac{\displaystyle\sum_{\substack{T_1 \subseteq E\left(G_{(\mathbf{r},w_1)}\right),\, T_1 \neq \emptyset \\ \cdots \\ T_\alpha \subseteq E\left(G_{(\mathbf{r},w_\alpha)}\right),\, T_\alpha \neq \emptyset}} m\left(\{T_1, \ldots, T_\alpha\}\right)}{\mu\left(E\left(G_{\mathbf{r}}\right)\right)}.$$

Proof. See Appendix. □

Example 2.2.1. *Let $\mathcal{F} = \left\{G_{(\mathbf{1},1)}\right\} \subseteq \mathcal{G}_{\mathbf{1}}^2$, as shown in Figure 2.3; considering the Möbius representation $m_{\mathbf{1}}^2$ of the capacity $\mu_{\mathbf{1}}^2$, we have that*

$$m_{\mathbf{1}}^2\left(\left\{G_{(\mathbf{1},1)}\right\}\right) = \frac{\displaystyle\sum_{T_1 \subseteq E\left(G_{(\mathbf{1},1)}\right),\, T_1 \neq \emptyset} m(\{T_1\})}{\mu\left(E\left(G_{\mathbf{1}}\right)\right)} =$$

$$= \frac{1}{\mu\left(E\left(G_{\mathbf{1}}\right)\right)}\left[m\left(\left\{g_{(\mathbf{1},1,1)}\right\}\right) + m\left(\left\{g_{(\mathbf{1},1,2)}\right\}\right) + m\left(\left\{g_{(\mathbf{1},1,1)}, g_{(\mathbf{1},1,2)}\right\}\right)\right].$$

Analogously, considering set $\mathcal{F} = \left\{G_{(\mathbf{1},1)}, G_{(\mathbf{1},2)}\right\} = \mathcal{G}_{\mathbf{1}}^2$, we have that

$$m_{\mathbf{1}}^2\left(\left\{G_{(\mathbf{1},1)}, G_{(\mathbf{1},2)}\right\}\right) = \sum_{\substack{T_1 \subseteq E\left(G_{(\mathbf{1},1)}\right),\, T_1 \neq \emptyset \\ T_2 \subseteq E\left(G_{(\mathbf{1},2)}\right),\, T_2 \neq \emptyset}} m\left(\{T_1, T_2\}\right) =$$

$$= \frac{1}{\mu\left(E\left(G_{\mathbf{1}}\right)\right)}\left[m\left(\left\{g_{(\mathbf{1},1,1)}, g_{(\mathbf{1},2,1)}\right\}\right) + m\left(\left\{g_{(\mathbf{1},1,1)}, g_{(\mathbf{1},2,2)}\right\}\right) + m\left(\left\{g_{(\mathbf{1},1,2)}, g_{(\mathbf{1},2,1)}\right\}\right) +\right.$$

$$+ m\left(\left\{g_{(\mathbf{1},1,2)}, g_{(\mathbf{1},2,2)}\right\}\right) + m\left(\left\{g_{(\mathbf{1},1,1)}, g_{(\mathbf{1},1,2)}, g_{(\mathbf{1},2,1)}+\right\}\right) + m\left(\left\{g_{(\mathbf{1},1,1)}, g_{(\mathbf{1},1,2)}, g_{(\mathbf{1},2,2)}\right\}\right) +$$

$$\left.+ m\left(\left\{g_{(\mathbf{1},1,1)}, g_{(\mathbf{1},2,1)}, g_{(\mathbf{1},2,2)}\right\}\right) + m\left(\left\{g_{(\mathbf{1},1,2)}, g_{(\mathbf{1},2,1)}, g_{(\mathbf{1},2,2)}\right\}\right)\right].$$

As mentioned before, 2-additive capacities are in general sufficient for practical use. For this reason, in the last part of this section we concentrate on the application of MCHP to the 2-additive Choquet integral preference model. First, we provide a proposition stating that if μ is a q-additive capacity, then $\mu_{\mathbf{r}}^k$ is also q-additive for each subcriterion $G_{\mathbf{r}}$ in the hierarchy, while the second proposition expresses the Shapley value and the interaction index in case of 2-additive capacities.

Proposition 2.2.2. *Let μ a q-additive capacity defined on 2^{EL}, then $\mu_{\mathbf{r}}^k$ is a q-additive capacity defined on $2^{\mathcal{G}_{\mathbf{r}}^k}$, for all $G_{\mathbf{r}} \in \mathcal{G}$ with $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus \{EL\}$.*

Proof. See Appendix. □

Proposition 2.2.3. *Let μ a 2-additive capacity defined on 2^{EL} and $G_{(\mathbf{r},w)}, G_{(\mathbf{r},w_1)}, G_{(\mathbf{r},w_2)} \in \mathcal{G}_{\mathbf{r}}^k$, with $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus \{EL\}$, then:*

1.

$$\varphi_{\mathbf{r}}^k\left(\left\{G_{(\mathbf{r},w)}\right\}\right) = \left[\sum_{\mathbf{t} \in E\left(G_{(\mathbf{r},w)}\right)} m\left(g_{\mathbf{t}}\right) + \sum_{\mathbf{t_1},\mathbf{t_2} \in E\left(G_{(\mathbf{r},w)}\right)} m\left(g_{\mathbf{t_1}}, g_{\mathbf{t_2}}\right) + \sum_{\substack{\mathbf{t_1} \in E\left(G_{(\mathbf{r},w)}\right) \\ \mathbf{t_2} \in E\left(\mathcal{G}_{\mathbf{r}}^k \setminus \{G_{(\mathbf{r},w)}\}\right)}} \frac{m(g_{\mathbf{t_1}}, g_{\mathbf{t_2}})}{2}\right] \frac{1}{\mu(E(G_{\mathbf{r}}))}, \tag{2.29}$$

2.

$$\varphi_{\mathbf{r}}^k\left(\left\{G_{(\mathbf{r},w_1)}, G_{(\mathbf{r},w_2)}\right\}\right) = \left[\sum_{\substack{\mathbf{t_1} \in E\left(G_{(\mathbf{r},w_1)}\right) \\ \mathbf{t_2} \in E\left(G_{(\mathbf{r},w_2)}\right)}} m(g_{\mathbf{t_1}}, g_{\mathbf{t_2}})\right] \frac{1}{\mu(E(G_{\mathbf{r}}))}. \tag{2.30}$$

Proof. See Appendix. □

It is meaningful observing in equation (2.29) that the importance of a criterion $G_{(\mathbf{r},w)}$ depends on which criterion it is descending from. This means that if $G_{(\mathbf{r},w)}$ is a subcriterion of $G_{\mathbf{r}}$ and, in turn, $G_{\mathbf{r}}$ is a subcriterion of $G_{\mathbf{s}}$, then $G_{(\mathbf{r},w)}$ will get importance $\varphi_{\mathbf{r}}^k(G_{(\mathbf{r},w)})$, because it is subcriterion of $G_{\mathbf{r}}$, and importance $\varphi_{\mathbf{s}}^k(G_{(\mathbf{r},w)})$, because it is also subcriterion of $G_{\mathbf{s}}$. This is due to the fact that when computing $\varphi_{\mathbf{s}}^k(G_{(\mathbf{r},w)})$ one should take into account a greater number of interactions than when computing $\varphi_{\mathbf{r}}^k(G_{(\mathbf{r},w)})$.

Example

In this section, we provide a simple example to explain how to apply the hierarchical Choquet integral preference model, and to highlight some characteristics of the method. Let us suppose that four alternative projects are evaluated with respect to two macro-criteria, Environmental (En) and Economic (Ec), and that each of these macro-criteria is composed of two elementary subcriteria. In particular, Soil Sustainability (SoSu) and Water Sustainability (WaSu) are elementary subcriteria of En while Expected Earnings (ExEa) and Financial Feasibility (FiFe) are elementary subcriteria of

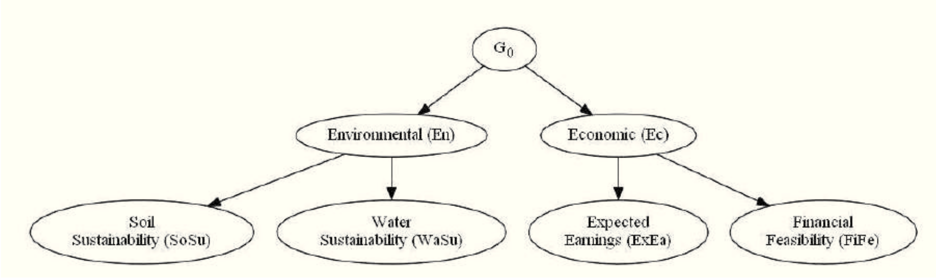


Figure 2.4: Hierarchy of criteria for evaluation of projects.

Ec. The hierarchy of criteria is shown in Figure 2.4. The general notation concerning set of criteria $\mathcal{G} = \left\{G_1, G_2, g_{(1,1)}, g_{(1,2)}, g_{(2,1)}, g_{(2,2)}\right\}$ corresponds now to $\mathcal{G} = \{$En,Ec,SoSu,WaSu,ExEa,FiFe$\}$. All criteria are defined on a common gain scale 10-30. The evaluations of the four projects with respect to the considered elementary subcriteria are shown in Table 2.8(a).

Table 2.8: Evaluations of projects and Möbius parameters

(a) Evaluations of projects

	En		Ec	
Projects	SoSu	WaSu	ExEa	FiFe
a	17	14	13	18
b	14	15	18	15
c	11	21	11	20
d	15	14	15	14

(b) Möbius parameters

m(SoSu)	0.3793
m(WaSu)	0.1724
m(ExEa)	0.0507
m(FiFe)	0.1562
m(SoSu, WaSu)	−0.1724
m(SoSu, ExEa)	−0.0507
m(SoSu, FiFe)	−0.1562
m(WaSu, ExEa)	0.6168
m(WaSu, FiFe)	0.0039
m(ExEa, FiFe)	0

For the sake of this example, we assume that the Möbius parameters of the Choquet integral are known (see Table 2.8(b)). They have been obtained by ordinal regression technique which will be explained in the next section. Values of the Choquet integral for the four projects can now be computed with respect to the totality of criteria (equation 2.18), as well as with respect to each of the two considered macro-criteria (equation 2.23). These values are given in Table 2.9. At the same time, one can also compute the Shapley values of different criteria (see Table 2.10).

Looking at Table 2.9, we can observe that even if project a is better than project b with respect to En and Ec ($C_{\mu_1}(a) > C_{\mu_1}(b)$ and $C_{\mu_2}(a) > C_{\mu_2}(b)$), b is preferred to a with respect to the totality of criteria ($C_\mu(b) > C_\mu(a)$). An analogous situation can be observed for projects c and d with c being preferred to d on the two macro-criteria and d being preferred to c with respect to the totality of criteria. Even if this could seem an unlikely situation at a first sight, it is justifiable if we observe

Table 2.9: Values of the Choquet integral for the four projects

(a) At the comprehensive level

	G_1 (**En**)		G_2 (**Ec**)		Choquet integral values
	SoSu	WaSu	ExEa	FiFe	$C_\mu(\cdot)$
a	17	14	13	18	$C_\mu(a) = 14.67$
b	14	15	18	15	$C_\mu(b) = 15.15$
c	11	21	11	20	$C_\mu(c) = 14.16$
d	15	14	15	14	$C_\mu(d) = 14.37$

(b) At the level of macro-criteria

	G_1 (**En**)		G_2 (**Ec**)		Choquet integral values
	SoSu	WaSu	ExEa	FiFe	$C_\mu(\cdot)/\mu(E(G_{\mathbf{r}}))$
a_1	17	14	0	0	$C_{\mu_1}(a) = 17$
a_2	0	0	13	18	$C_{\mu_2}(a) = 16.77$
b_1	14	15	0	0	$C_{\mu_1}(b) = 14.45$
b_2	0	0	18	15	$C_{\mu_2}(b) = 15.73$
c_1	11	21	0	0	$C_{\mu_1}(c) = 15.54$
c_2	0	0	11	20	$C_{\mu_2}(c) = 17.79$
d_1	15	14	0	0	$C_{\mu_1}(d) = 15$
d_2	0	0	15	14	$C_{\mu_2}(d) = 14.24$

that in computing the Choquet integral of a project with respect to En (analogously with respect to Ec), we take into account only the interactions between the elementary subcriteria of En (Ec), while in computing the Choquet integral of a project with respect to the totality of criteria, we consider the interactions between all four elementary subcriteria.

Table 2.10: Shapley values

(a) Shapley values of each elementary subcriterion with respect to the considered macro-criterion

	En		**Ec**	
	SoSu	WaSu	ExEa	FiFe
$\varphi^2_{\mathbf{r}}(G_{(\mathbf{r},w)})$	0.7727	0.2272	0.2450	0.7549

(b) Shapley values of each elementary subcriterion with respect to the totality of criteria

	$\varphi^2_{\mathbf{0}}(G_{(\mathbf{r},w)})$
SoSu	0.1896
WaSu	0.3965
ExEa	0.3337
FiFe	0.080

Looking at Table 2.10, one can observe another phenomenon characteristic for the hierarchical Choquet integral preference model. Indeed, SoSu is more important than WaSu when they are considered as subcriteria of En, while the opposite is true when they are considered as subcriteria of the root criterion G_0. Analogous situation could be observed for the elementary subcriteria ExEa and FiFe, where FiFe is more important than ExEa when they are considered as subcriteria of Ec, while the opposite is true when they are considered as elementary subcriteria of the root criterion G_0. Also this phenomenon could seem unlikely, but this can be explained as before. In fact, when computing the importance of SoSu with respect to En one has to take into account only the interaction between SoSu and WaSu, while computing the importance of SoSu with respect to the root criterion G_0, one has to take into account its interaction with WaSu and the two elementary subcriteria of Ec.

2.2.3 Robust Ordinal Regression (ROR) and Stochastic Multiobjective Acceptability Analysis (SMAA) applied to the hierarchical Choquet integral preference model

According to Section 2.2.2, to apply the hierarchical Choquet integral preference model, one has to define the Möbius representation of a capacity defined on the power set of EL, that is $m(\{g_{\mathbf{t}}\})$ for each elementary subcriterion $g_{\mathbf{t}}$, and $m(\{g_{\mathbf{t_1}}, g_{\mathbf{t_2}}\})$ for each couple of elementary subcriteria $\{g_{\mathbf{t_1}}, g_{\mathbf{t_2}}\}$. These values will be calculated using an ordinal regression technique from some indirect preference information. Below, we explain this technique in detail.

Given a criterion $G_{\mathbf{r}}$, $\mathbf{r} \in \mathcal{I}_{\mathcal{G}} \setminus \{EL\}$, the DM is requested to provide the following type of preference information:

- a is preferred to b on criterion $G_{\mathbf{r}}$, denoted by $a \succ_{\mathbf{r}} b$ (translated to the constraint $C_{\mu_{\mathbf{r}}}(a) \geq C_{\mu_{\mathbf{r}}}(b) + \varepsilon$);
- a is indifferent to b on criterion $G_{\mathbf{r}}$, denoted by $a \sim_{\mathbf{r}} b$ ($C_{\mu_{\mathbf{r}}}(a) = C_{\mu_{\mathbf{r}}}(b)$);
- on criterion $G_{\mathbf{r}}$, a is preferred to b more than c is preferred to d, denoted by $(a, b) \succ^*_{\mathbf{r}} (c, d)$, ($C_{\mu_{\mathbf{r}}}(a) - C_{\mu_{\mathbf{r}}}(b) \geq C_{\mu_{\mathbf{r}}}(c) - C_{\mu_{\mathbf{r}}}(d) + \varepsilon$ and $C_{\mu_{\mathbf{r}}}(c) - C_{\mu_{\mathbf{r}}}(d) \geq \varepsilon$);
- on criterion $G_{\mathbf{r}}$ the intensity of preference between a and b is the same as the intensity of preference between c and d, denoted by $(a, b) \sim^*_{\mathbf{r}} (c, d)$ ($C_{\mu_{\mathbf{r}}}(a) - C_{\mu_{\mathbf{r}}}(b) = C_{\mu_{\mathbf{r}}}(c) - C_{\mu_{\mathbf{r}}}(d)$).

Considering criteria $G_{\mathbf{r_1}}, G_{\mathbf{r_2}}, G_{\mathbf{r_3}}, G_{\mathbf{r_4}}$, with $\mathbf{r_1}, \mathbf{r_2}, \mathbf{r_3}, \mathbf{r_4} \in \mathcal{G}^k_{\mathbf{r}}$, the DM can provide the following preference information:

- criterion $G_{\mathbf{r_1}}$ is more important than criterion $G_{\mathbf{r_2}}$, denoted by $G_{\mathbf{r_1}} \succ G_{\mathbf{r_2}}$ (translated to the constraint $\varphi^k_{\mathbf{r}}(\{G_{\mathbf{r_1}}\}) \geq \varphi^k_{\mathbf{r}}(\{G_{\mathbf{r_2}}\}) + \varepsilon$);
- criteria $G_{\mathbf{r_1}}$ and $G_{\mathbf{r_2}}$ are equally important, denoted by $G_{\mathbf{r_1}} \sim G_{\mathbf{r_2}}$ ($\varphi^k_{\mathbf{r}}(\{G_{\mathbf{r_1}}\}) = \varphi^k_{\mathbf{r}}(\{G_{\mathbf{r_2}}\})$);
- criteria $G_{\mathbf{r_1}}$ and $G_{\mathbf{r_2}}$ are positively interacting ($\varphi^k_{\mathbf{r}}(\{G_{\mathbf{r_1}}, G_{\mathbf{r_2}}\}) \geq \varepsilon$);
- criteria $G_{\mathbf{r_1}}$ and $G_{\mathbf{r_2}}$ are negatively interacting ($\varphi^k_{\mathbf{r}}(\{G_{\mathbf{r_1}}, G_{\mathbf{r_2}}\}) \leq -\varepsilon$);
- the interaction between criteria $G_{\mathbf{r_1}}$ and $G_{\mathbf{r_2}}$ is greater than the interaction between criteria $G_{\mathbf{r_3}}$ and $G_{\mathbf{r_4}}$
  - if there is positive interaction between both pairs of criteria, then the constraint translating this preference are $\varphi^k_{\mathbf{r}}(\{G_{\mathbf{r_1}}, G_{\mathbf{r_2}}\}) - \varphi^k_{\mathbf{r}}(\{G_{\mathbf{r_3}}, G_{\mathbf{r_4}}\}) \geq \varepsilon$ and $\varphi^k_{\mathbf{r}}(\{G_{\mathbf{r_3}}, G_{\mathbf{r_4}}\}) \geq \varepsilon$

– if there is negative interaction between both pairs of criteria, then the constraint translating this preference are $\varphi^k_{\mathbf{r}}(\{G_{\mathbf{r_1}}, G_{\mathbf{r_2}}\}) - \varphi^k_{\mathbf{r}}(\{G_{\mathbf{r_3}}, G_{\mathbf{r_4}}\}) \leq -\varepsilon$ and $\varphi^k_{\mathbf{r}}(\{G_{\mathbf{r_3}}, G_{\mathbf{r_4}}\}) \leq -\varepsilon$;

- $G_{\mathbf{r_1}}$ is preferred to $G_{\mathbf{r_2}}$ more than $G_{\mathbf{r_3}}$ is preferred to $G_{\mathbf{r_4}}$, denoted by $(G_{\mathbf{r_1}}, G_{\mathbf{r_2}}) \succ^* (G_{\mathbf{r_3}}, G_{\mathbf{r_4}})$ $\big(\varphi^k_{\mathbf{r}}(\{G_{\mathbf{r_1}}\}) - \varphi^k_{\mathbf{r}}(\{G_{\mathbf{r_2}}\}) \geq \varphi^k_{\mathbf{r}}(\{G_{\mathbf{r_3}}\}) - \varphi^k_{\mathbf{r}}(\{G_{\mathbf{r_4}}\}) + \varepsilon$ and $\varphi^k_{\mathbf{r}}(\{G_{\mathbf{r_3}}\}) - \varphi^k_{\mathbf{r}}(\{G_{\mathbf{r_4}}\}) \geq \varepsilon\big)$;

- the difference of importance between $G_{\mathbf{r_1}}$ and $G_{\mathbf{r_2}}$ is the same of the difference of importance between $G_{\mathbf{r_3}}$ and $G_{\mathbf{r_4}}$, denoted by $(G_{\mathbf{r_1}}, G_{\mathbf{r_2}}) \sim^* (G_{\mathbf{r_3}}, G_{\mathbf{r_4}})$ $(\varphi^k_{\mathbf{r}}(\{G_{\mathbf{r_1}}\}) - \varphi^k_{\mathbf{r}}(\{G_{\mathbf{r_2}}\}) = \varphi^k_{\mathbf{r}}(\{G_{\mathbf{r_3}}\}) - \varphi^k_{\mathbf{r}}(\{G_{\mathbf{r_4}}\}))$.

Similarly to Section 2.2.2, ε is an auxiliary variable used to convert the strict inequalities into weak ones. Moreover, like in Section 2.2.2, E^{DM} denotes the set of constraints translating the DM's preference information together with the monotonicity and normalization constraints.
To check if there exists at least one compatible capacity, one has to solve the following linear programming problem:

$$\varepsilon^* = \max \varepsilon, \text{ subject to } E^{DM}.$$

If E^{DM} is feasible, and $\varepsilon^* > 0$ then there exists at least one compatible capacity, otherwise some inconsistency arised, which has to be identified [125].
Considering criterion $G_{\mathbf{r}}$ located at a not last level of the hierarchy, and the two following sets of constraints,

$$\left.\begin{array}{l} C_{\mu_{\mathbf{r}}}(b) \geq C_{\mu_{\mathbf{r}}}(a) + \varepsilon, \\ E^{DM}. \end{array}\right\} E^N_{\mathbf{r}}(a,b), \qquad \left.\begin{array}{l} C_{\mu_{\mathbf{r}}}(a) \geq C_{\mu_{\mathbf{r}}}(b) \\ E^{DM} \end{array}\right\} E^P_{\mathbf{r}}(a,b)$$

the necessary preference relation with respect to criterion $G_{\mathbf{r}}$ holds for alternatives a and b if $E^N_{\mathbf{r}}(a,b)$ is infeasible or $\varepsilon^N_{\mathbf{r}} \leq 0$, where $\varepsilon^N_{\mathbf{r}} = \max \varepsilon$, subject to $E^N_{\mathbf{r}}(a,b)$. Analogously, the possible preference relation with respect to criterion $G_{\mathbf{r}}$ holds for alternatives a and b if $E^P_{\mathbf{r}}(a,b)$ is feasible and $\varepsilon^P_{\mathbf{r}} > 0$, where $\varepsilon^P_{\mathbf{r}} = \max \varepsilon$, subject to $E^P_{\mathbf{r}}(a,b)$.

In practice, it is very likely that, given an available preference information, a is possibly preferred to b and b is possibly preferred to a. Nevertheless, the number of compatible capacities for which a is preferred to b could be very different from the number of compatible capacities for which b is preferred to a. For this reason, in order to estimate how good an alternative is compared to others and how often it is preferred over another alternative, we propose to apply the SMAA methodology. This technique applied to the hierarchical Choquet integral preference model is explained in detail below.

The set of linear constraints in E^{DM} defines a convex set of Möbius parameters. To explore this set of parameters the Hit-And-Run (HAR) method can be applied [158, 168, 172]. HAR samples iteratively a set of Möbius parameters satisfying E^{DM} until a stopping condition is met. For each sampled set of Möbius parameters and a given criterion $G_{\mathbf{r}}$, one can compute values of the Choquet integral for all considered alternatives. These values rank the alternatives with respect to $G_{\mathbf{r}}$. Having as many rankings as the samples, one can compute the indices typical to the SMAA methodology recalled in Section 2.2.2:

- the rank acceptability index $b^{l}_{k,\mathbf{r}}$, being the frequency with which alternative a_k gets position l in the ranking obtained with respect to criterion $G_{\mathbf{r}}$,
- the pairwise winning index $p_{\mathbf{r}}(a,b)$, giving the frequency of the preference of a over b on criterion $G_{\mathbf{r}}$.

Moreover, by using the rank acceptability indices, other two indices recently introduced in [2] can be computed:

- the downward cumulative rank acceptability index $b^{\leq l}_{k,\mathbf{r}}$, being the frequency that alternative a_k will get a position not greater than l on criterion $G_{\mathbf{r}}$,

$$b^{\leq l}_{k,\mathbf{r}} = \sum_{s=1}^{l} b^{s}_{k,\mathbf{r}},$$

- the upward cumulative rank acceptability index $b^{\geq l}_{k,\mathbf{r}}$, being the frequency that alternative a_k will get a position not lower than l on criterion $G_{\mathbf{r}}$,

$$b^{\geq l}_{k,\mathbf{r}} = \sum_{s=l}^{n} b^{s}_{k,\mathbf{r}}.$$

It is worth stressing that at the comprehensive level, represented by criterion $G_{\mathbf{0}}$, we also get the necessary and possible preference relations on one hand and the SMAA indices on the other hand.

An illustrative real world decision making problem

In this section, we apply the proposed methodology to a real world decision making problem [1]. 220 European universities from 30 countries have been evaluated on a 1-5 scale (1-weak, 2-below average, 3-average, 4-good, 5-very good) with respect to criteria structured in a hierarchical way, as shown in Figure 2.5. The three macro-criteria are Teaching & Learning (TL), Research (R) and

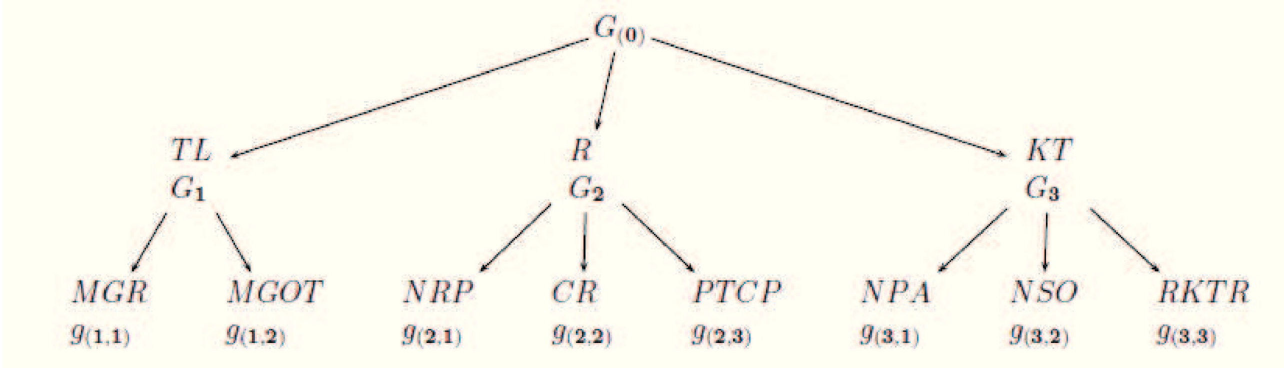


Figure 2.5: Hierarchical structure of criteria considered in the case study

Knowledge Transfer (KT), and they are further decomposed to more detailed elementary subcriteria. For macro-criteron TL, these are:

- Masters Graduation Rate (MGR),
- Masters Graduating on Time (MGOT).

Macro-criterion R is decomposed to:

- Number of Research Publications (NRP),
- Citation Rate (CR),
- Proportion of Top Cited Publications (PTCT),

and macro-criterion KT is decomposed to:

- Number of Patents Awarded (NPA),
- Number of Spin-Offs (NSO),
- Research and Knowledge Transfer Revenues (RKTR).

Description of the elementary subcriteria is given in Table 2.11.

Many of these universities dominate[4] the others and, at the same time, many universities are dominated by others. For this reason, following a procedure well known from the evolutionary multiobjective optimization method, called NSGA-II [39], we ordered the universities in nondominated fronts. We put in the first front all nondominated universities; then, after removing these universities from the list of universities, we put in the second front the universities nondominated among the

[4] An alternative a dominates an alternative b with respect to criteria $\{g_1, \dots, g_n\}$ if, supposing that all criteria are of the gain type, $g_i(a) \geq g_i(b)$ for all $i = 1, \dots, n$, and there exists at least one $j \in \{1, \dots, n\}$, such that $g_j(a) > g_j(b)$.

Table 2.11: Description of the elementary subcriteria

Elementary subriterion	Description
Masters Graduation Rate (MGR)	The percentage of new entrants that successfully completed their master programs
Masters Graduating on Time (MGOT)	The percentage of graduates that graduated within the time expected (normative time) for their masters programs
Number of Research Publications (NRP)	The number of research publications indexed in the Web of Science database, where at least one author is affiliated to the university (relative to the number of students)
Citation Rate (CR)	The average number of times that the university department's research publications (over the period 2008-2011) get cited in other research, adjusted (normalized) at the global level to take into account differences in publication years and to allow for differences
Proportion of Top Cited Publications (PTCP)	The proportion of the university's research publications that, compared to other publications in the same field and in the same year, belong to the top 10% most frequently cited
Number of Patents Awarded (NPA)	The number of patents assigned to (inventors working in) the university (over the period 2001-2010)
Number of Spin-Offs (NSO)	The number of spin-offs (i.e. firms established on the basis of a formal knowledge transfer arrangement between the institution and the firm) recently created by the institution (per 1,000 fte academic staff)
Research and Knowledge Transfer Revenues (RKTR)	Research revenues and knowledge transfer revenues from private sources (incl. not-for profit organizations), excluding tuition fees. Measured in €1,000s using Purchasing Power Parities. Expressed per fte academic staff.

remaining ones, and so on. In this way, the universities belonging to the same front are more ore less similar, in the sense that there is not any strong evidence for the preference of one university over another. Consequently, it is meaningful from the DM's point of view to get a ranking recommendation with respect to universities from the same front. In this section, we shall focus our attention on the first nondominated front but, of course, a similar analysis could be done also with respect to another nondominated front, or with respect to any subset of universities considered as most interesting for a particular DM. The evaluations of the universities belonging to the first nondominated front on the eight elementary subcriteria are provided in Table 2.12.

Table 2.12: Evaluations of the universities belonging to the first nondominated front on the considered elementary subcriteria

		$G_{(\mathbf{0})}$							
		TL ($G_{(\mathbf{1})}$)		**R** ($G_{(\mathbf{2})}$)			**KT** ($G_{(\mathbf{3})}$)		
University	**Country**	**MGR** ($g_{(\mathbf{1},1)}$)	**MGOT** ($g_{(\mathbf{1},2)}$)	**NRP** ($g_{(\mathbf{2},1)}$)	**CR** ($g_{(\mathbf{2},2)}$)	**PTCP** ($g_{(\mathbf{2},3)}$)	**NPA** ($g_{(\mathbf{3},1)}$)	**NS0** ($g_{(\mathbf{3},2)}$)	**RKTR** ($g_{(\mathbf{3},3)}$)
Bocconi University (U_{25})	Italy	5	4	2	5	5	1	1	5
Budapest U Tech & Economics (U_{35})	Hungary	5	3	3	3	3	2	4	2
U Cordoba (U_{51})	Spain	3	5	3	3	3	2	3	5
Tech U Denmark (U_{61})	Denmark	4	4	5	5	5	5	5	5
Dublin Inst. Tech (U_{64})	Ireland	2	5	2	5	5	2	4	2
U Limerick (U_{108})	Ireland	4	5	2	5	4	4	3	5
Lomonosow Moscow State U (U_{117})	Russia	5	5	5	2	2	2	5	5
Mondragon U (U_{129})	Spain	4	5	2	5	5	1	5	5
Newcastle U (U_{136})	United Kingdom	4	5	5	5	5	5	2	5
U Salamanca (U_{170})	Spain	5	4	4	3	3	2	2	4
U Trieste (U_{196})	Italy	5	2	5	4	4	3	3	3
WHU School of Management (U_{216})	Germany	5	5	2	4	4	1	5	5

Suppose that the DM specifies the following preference information on the considered elementary subcriteria and on the macro-criteria. Within parentheses, we write the constraints translating the corresponding piece of preference information provided by the DM:

- R is more important than KT that, in turn, is more important than TL
  ($\varphi_{\mathbf{0}}(R) \geq \varphi_{\mathbf{0}}(KT) + \varepsilon$ and $\varphi_{\mathbf{0}}(KT) \geq \varphi_{\mathbf{0}}(TL) + \varepsilon$),

- With respect to TL, MGOT is more important than MGR ($\varphi^2_{\mathbf{2}}(\{MGOT\}) \geq \varphi^2_{\mathbf{2}}(\{MGR\}) + \varepsilon$),

- With respect to KT, RKTR is more important than NSO that, in turn, is more important than NPA ($\varphi^2_{\mathbf{2}}(\{RKTR\}) \geq \varphi^2_{\mathbf{2}}(\{NSO\}) + \varepsilon$ and $\varphi^2_{\mathbf{2}}(\{NSO\}) \geq \varphi^2_{\mathbf{2}}(\{NPA\}) + \varepsilon$),

- At a comprehensive level, PTCP is more important than RKTR that, in turn, is more important than MGT ($\varphi^2_{\mathbf{0}}(\{PTCP\}) \geq \varphi^2_{\mathbf{0}}(\{RKTR\}) + \varepsilon$ and $\varphi^2_{\mathbf{0}}(\{RKTR\}) \geq \varphi^2_{\mathbf{0}}(\{MGT\}) + \varepsilon$),

- TL and R are positively interacting ($\varphi^1_{\mathbf{0}}(\{TL, R\}) \geq \varepsilon$),

- R and KT are positively interacting ($\varphi^1_{\mathbf{0}}(\{R, KT\}) \geq \varepsilon$),

- TL and KT are positively interacting ($\varphi^1_{\mathbf{0}}(\{TL, KT\}) \geq \varepsilon$),

- The interaction between R and KT is greater than the interaction between TL and KT ($\varphi^1_{\mathbf{0}}(\{R, KT\}) \geq \varphi^1_{\mathbf{0}}(\{TL, KT\}) + \varepsilon$ and $\varphi^1_{\mathbf{0}}(\{TL, KT\}) \geq \varepsilon$),

- The interaction between R and TL is greater than the interaction between TL and KT ($\varphi^1_{\mathbf{0}}(\{R, TL\}) \geq \varphi^1_{\mathbf{0}}(\{TL, KT\}) + \varepsilon$ and $\varphi^1_{\mathbf{0}}(\{TL, KT\}) \geq \varepsilon$),

- With respect to R, NRP and PTCP are positively interacting ($\varphi^2_{\mathbf{2}}(\{NRP, PTCP\}) \geq \varepsilon$),

- CR and PTCP are negatively interacting ($\varphi^2_{\mathbf{0}}(\{CR, PTCP\}) \leq -\varepsilon$),

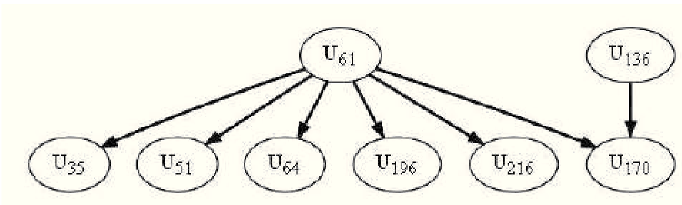
- NRP and RKTR are positively interacting ($\varphi^2_{\mathbf{0}}(\{NRP, RKTR\}) \geq \varepsilon$),

- NPA and NSO are negatively interacting ($\varphi^2_{\mathbf{0}}(\{NPA, NSO\}) \leq -\varepsilon$),

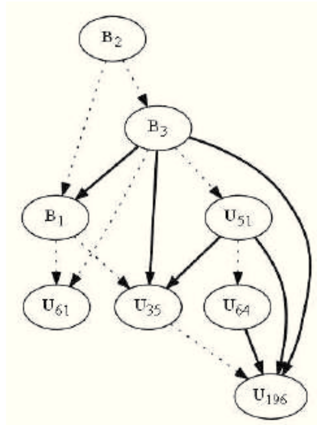
- MGOT and NRP are positively interacting ($\varphi^2_{\mathbf{0}}(\{MGOT, NRP\}) \geq \varepsilon$),

Applying NAROR at the comprehensive level, as well as on the three macro-criteria, we get the necessary preference relations shown in Figures 2.6(a)-2.6(d). Let us observe that the blocks $B_1, \ldots, B_7$ in Figures 2.6(b)-2.6(d) are composed of universities having exactly the same evaluations on the elementary subcriteria descending from the considered macro-criterion. Therefore, for example, B_1 is composed of U_{25} and U_{170} since they have exactly the same evaluations (5 and 4) on MGR and MGOT, being the two elementary subcriteria descending from macro-criterion TL.

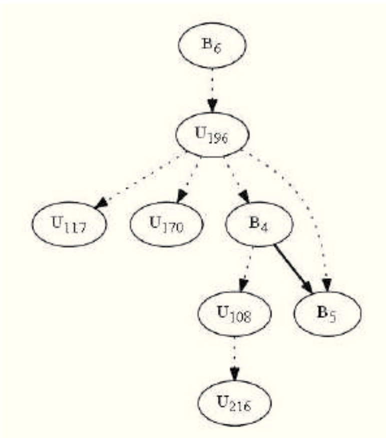
Looking at Figures 2.6(a)-2.6(d) it seems that U_{61} can be seen as the best university. Indeed, while it is evident that on R and KT this university dominates all the others, at the comprehensive level it is necessarily preferred to six out of the eleven universities. Analyzing more in detail the results of NAROR at the intermediate level, one can observe that the preference information provided by the DM result in many bold arrows, i.e., necessary preference relations which are not dominance



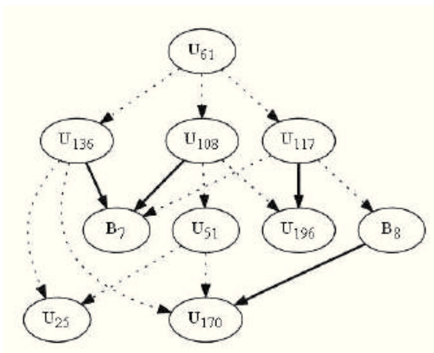
(a) Comprehensive level



(b) Teaching and Learning (TL); $B_1 = \{U_{25}, U_{170}\}$, $B_2 = \{U_{117}, U_{216}\}$, $B_3 = \{U_{108}, U_{129}, U_{136}\}$



(c) Research (R); $B_4 = \{U_{25}, U_{64}, U_{129}\}$, $B_5 = \{U_{35}, U_{51}\}$, $B_6 = \{U_{61}, U_{136}\}$



(d) Knowledge Transfer (KT); $B_7 = \{U_{35}, U_{64}\}$, $B_8 = \{U_{129}, U_{216}\}$

Figure 2.6: Necessary preference relation at the comprehensive level, as well as with respect to macro-criteria TL, R and KT. Dotted arrows represent the dominance relation, while bold arrows represent necessary preference relations obtained by NAROR.

relations. For example, on TL, U_{51} is necessarily preferred to U_{35} and U_{196}, while on R, the universities belonging to B_4 are necessarily preferred to the universities belonging to B_5. Moreover, on KT, U_{117} is necessarily preferred to U_{196}. Let us remind that the results we are showing concern the universities belonging to the first nondominated front only but they are enough to observe that the application of NAROR puts many new couples in the necessary preference relations, both at the comprehensive level, and on particular macro-criteria, contributing in this way to a better understanding of the decision problem by the DM.

After applying NAROR, we applied the SMAA methodology on the set of compatible value functions at the comprehensive level and at the level of macro-criteria. At first, for each considered university, we looked at the best and at the worst position the university could get considering the whole set of capacities compatible with the preferences provided by the DM as well as the three highest rank acceptability indices showing, therefore, which are the most likely positions for the alternatives at hand.

Table 2.13: Rank Acceptability Indices. For each university, we reported the worst and the best possible positions as well as the three greatest rank acceptability indices. The blocks $B_1, \ldots, B_8$ are composed of universities having exactly the same evaluations. Let us note that 220 are the different performance vectors got by the universities at comprehensive level so the rank acceptability indices are computed for the positions going from the first one to the 220th. Analogously, the rank acceptability indices with respect to TL are computed for the positions from the first to the 16th, with respect to R from the first to the 29th while, on KT, from the first to the 55th.

(a) Comprehensive level

University	Best $\left(b_{k,\mathbf{0}}^{Best}\right)$	Worst $\left(b_{k,\mathbf{0}}^{Worst}\right)$	$high_1\left(b_{k,\mathbf{0}}^{high_1}\right)$	$high_2\left(b_{k,\mathbf{0}}^{high_2}\right)$	$high_3\left(b_{k,\mathbf{0}}^{high_3}\right)$
U_{25}	18 (0.11%)	103 (0.01%)	52 (3.65%)	55 (3.45%)	58 (3.39%)
U_{35}	80 (0.01%)	132 (0.05%)	106 (4.68%)	108 (4.04%)	105 (3.87%)
U_{51}	47 (0.1%)	83 (0.01%)	65 (6.18%)	60 (6%)	70 (5.71%)
U_{61}	1 (93.29%)	2 (6.71%)			
U_{64}	47 (0.02%)	133 (0.05%)	90 (2.82%)	87 (2.72%)	88 (2.66%)
U_{108}	14 (0.16%)	51 (0.01%)	28 (8.55%)	34 (6.19%)	33 (5.8%)
U_{117}	23 (0.15%)	90 (0.06%)	63 (3.22%)	67 (3.15%)	64 (2.82%)
U_{129}	6 (0.2%)	68 (0.11%)	23 (4.8%)	32 (4.24%)	37 (4.12%)
U_{136}	1 (6.71%)	12 (0.12%)	2 (60.18%)	3 (14.36%)	5 (8.88%)
U_{170}	58 (0.04%)	99 (0.04%)	80 (6.04%)	82 (6.12%)	83 (7.02%)
U_{196}	40 (0.01%)	77 (0.33%)	58 (5.88%)	57 (5.64%)	55 (5.6%)
U_{216}	12 (0.04%)	81 (0.01%)	43 (4.59%)	46 (4.04%)	42 (3.65%)

(b) Teaching and Learning

University	Best $\left(b_{k,\mathbf{1}}^{Best}\right)$	Worst $\left(b_{k,\mathbf{1}}^{Worst}\right)$	$high_1\left(b_{k,\mathbf{1}}^{high_1}\right)$	$high_2\left(b_{k,\mathbf{1}}^{high_2}\right)$	$high_3\left(b_{k,\mathbf{1}}^{high_3}\right)$
B_1	3 (61.91%)	5 (18.47%)	3 (61.91%)	4 (19.62%)	5 (18.47%)
B_2	1 (100.00%)	1 (100.00%)			
B_3	2 (100.00%)	2 (100.00%)			
U_{35}	5 (5.93%)	9 (19.37%)	6 (32.95%)	8 (28.3%)	9 (19.37%)
U_{51}	3 (38.09%)	5 (21.64%)	4 (40.27%)	3 (38.09%)	5 (21.64%)
U_{61}	4 (21.64%)	7 (4.04%)	6 (37.63%)	5 (36.69%)	4 (21.64%)
U_{64}	4 (18.47%)	10 (0.05%)	7 (23.9%)	6 (23.52%)	4 (18.47%)
U_{196}	8 (4.48%)	13 (28.09%)	13 (28.09%)	12 (22.16%)	10 (21.72%)

(c) Research

University	Best $\left(b_{k,\mathbf{1}}^{Best}\right)$	Worst $\left(b_{k,\mathbf{1}}^{Worst}\right)$	$high_1\left(b_{k,\mathbf{1}}^{high_1}\right)$	$high_2\left(b_{k,\mathbf{1}}^{high_2}\right)$	$high_3\left(b_{k,\mathbf{1}}^{high_3}\right)$
B_4	5 (6.9%)	16 (1.27%)	8 (22.16%)	10 (13.81%)	11 (13.72%)
B_5	15 (1.28%)	23 (0.14%)	21 (24.59%)	18 (20.95%)	20 (20.65%)
B_6	1 (100.00%)	1 (100.00%)			
U_{108}	6 (0.32%)	19 (4.23%)	13 (16.05%)	14 (15.25%)	12 (11.63%)
U_{117}	20 (1.71%)	26 (29.18%)	25 (48.64%)	26 (29.18%)	24 (14.85%)
U_{170}	12 (3.03%)	21 (1.34%)	18 (27.28%)	17 (21.32%)	19 (16.49%)
U_{196}	4 (43.99%)	10 (0.01%)	4 (43.99%)	5 (32.95%)	6 (10.96%)
U_{216}	14 (4.66%)	21 (1.71%)	16 (23.79%)	15 (18.16%)	19 (17.19%)

(d) Knowledge Transfer

University	Best $\left(b_{k,\mathbf{1}}^{Best}\right)$	Worst $\left(b_{k,\mathbf{1}}^{Worst}\right)$	$high_1\left(b_{k,\mathbf{1}}^{high_1}\right)$	$high_2\left(b_{k,\mathbf{1}}^{high_2}\right)$	$high_3\left(b_{k,\mathbf{1}}^{high_3}\right)$
B_7	30 (0.18%)	42 (0.3%)	36 (21.38%)	35 (18.18%)	34 (15.78%)
B_8	15 (1.28%)	23 (0.14%)	21 (24.59%)	18 (20.95%)	20 (20.65%)
U_{25}	21 (0.09%)	40 (0.09%)	30 (25.86%)	29 (11.91%)	31 (10.55%)
U_{51}	13 (1.04%)	20 (0.38%)	16 (30.46%)	17 (22.81%)	18 (21.49%)
U_{61}	1 (100.00%)	1 (100.00%)			
U_{108}	7 (4.92%)	14 (0.34%)	8 (20.09%)	10 (19.71%)	11 (17.92%)
U_{117}	4 (20.81%)	14 (0.32%)	7 (22.22%)	4 (20.81%)	5 (19.14%)
U_{136}	4 (5.2%)	20 (3.17%)	7 (11.48%)	10 (10.91%)	5 (9.05%)
U_{170}	25 (0.3%)	36 (0.5%)	31 (34.4%)	32 (29.06%)	30 (9.87%)
U_{196}	26 (0.01%)	39 (0.77%)	36 (17.57%)	33 (16.3%)	35 (15.43%)

Looking at Tables 2.13(a)-2.13(d) the following considerations could be done:

- At comprehensive level, U_{61} is confirmed as the best among the considered universities since it has rank acceptability index for the 1st position equal to 93.29% while the remaining 6.71% is its rank acceptability index for the 2nd position; analogously, U_{136} could be considered a really good university since it fills always a position between the 1st and the 12th and its highest rank acceptability indices are those corresponding to the 2nd and to the 3rd positions. At the same time, even if U_{35} and U_{64} belong to the highest nondominated front, they do not fill very high positions in the complete rankings obtained at a comprehensive level. Indeed, on one hand, the highest position reached by U_{35} is the 80th while its highest rank acceptability indices corresponds to the position 106. On the other hand, U_{64} reaches positions between the 47th and the 133th and its highest rank acceptability index is obtained in correspondence of the position 90.

- With respect to TL the complete ranking is almost sure. Indeed, the universities belonging to the block B_1, that are U_{25} and U_{170}, are always in the 1st position while the universities belonging to the block B_2, that are U_{117} and U_{216}, are always in the 2nd position. Considering that on this macro-criterion the possible positions are only sixteen since sixteen different performance vectors are obtained by the considered universities on this macro-criterion, U_{196} is bad on this macro-criterion since it fills always a position between the 8th and the 13th and its highest rank acceptability index is obtained in correspondence of the position 13.

- With respect to R, the universities belonging to the block B_4, that are U_{25}, U_{64} and U_{129}, are the best since they fill always the 1st position. Good results are also obtained by university U_{196} which fills always positions between the 4th and the 10th and it has the highest rank acceptability index in correspondence of the 4th position. U_{117} is instead a bad university on this macro-criterion since it fills positions between the 20th and the 26th and its highest rank acceptability index is obtained in correspondence of the position 25.

- On KT, U_{61} is always the first while U_{117} and U_{136} are quite good since the highest position got by both of them is the 4th and their greatest rank acceptability index is obtained in correspondence of the 7th position. At the same time, the universities belonging to block B_7, that are U_{35} and U_{64}, are not very good on KT since they fill always a position between the 30th and the 42th but their highest rank acceptability index is got in correspondence of the position 36.

To compare the universities pairwise, we computed also the pairwise winning indices $p(U_h, U_k)$ providing the frequency with which university U_h is preferred to university U_k considering all criteria simultaneously, that is at comprehensive level, as well as considering the three macro-criteria singularly.

Table 2.14: Pairwise Winning Indices

(a) Comprehensive level

$p_{\mathbf{0}}(\cdot,\cdot)$	U_{25}	U_{35}	U_{51}	U_{61}	U_{64}	U_{108}	U_{117}	U_{129}	U_{136}	U_{170}	U_{196}	U_{216}
U_{25}	0	99.87	77.21	0	94.75	2.69	55.06	3.36	0	94.93	54.78	19.78
U_{35}	0.13	0	0	0	28.34	0	0	0	0	1.13	0	0
U_{51}	22.79	100	0	0	88.08	0	32.1	0.82	0	96.44	23.16	4.65
U_{61}	100	100	100	0	100	100	100	100	93.29	100	100	100
U_{64}	5.25	71.66	11.92	0	0	0	9.85	0	0	26.51	2.25	2.67
U_{108}	97.31	100	100	0	100	0	96.16	47.92	0	100	99.67	86.38
U_{117}	44.94	100	67.9	0	90.15	3.84	0	8.82	0	96.72	45.94	21.63
U_{129}	96.64	100	99.18	0	100	52.08	91.18	0	0.24	100	90.22	99.95
U_{136}	100	100	100	6.71	100	100	100	99.76	0	100	100	100
U_{170}	5.07	98.87	3.56	0	73.49	0	3.28	0	0	0	1.33	0.3
U_{196}	45.22	100	76.84	0	97.75	0.33	54.06	9.78	0	98.67	0	18.88
U_{216}	80.22	100	95.35	0	97.33	13.62	78.37	0.05	0	99.7	81.12	0

(b) Teaching and Learning

$p_{\mathbf{1}}(\cdot,\cdot)$	B_1	B_2	B_3	U_{35}	U_{51}	U_{61}	U_{64}	U_{196}
B_1	0	0	0	100	61.91	100	81.53	100
B_2	100	0	100	100	100	100	100	100
B_3	100	0	0	100	100	100	100	100
U_{35}	0	0	0	0	0	8.97	36.94	100
U_{51}	38.09	0	0	100	0	78.36	100	100
U_{61}	0	0	0	91.03	21.64	0	63.26	100
U_{64}	18.47	0	0	63.06	0	36.74	0	100
U_{196}	0	0	0	0	0	0	0	0

(c) Research

$p_{\mathbf{2}}(\cdot,\cdot)$	B_4	B_5	B_6	U_{108}	U_{117}	U_{170}	U_{196}	U_{216}
B_4	0	98.73	0	100	100	94.37	14.74	100
B_5	1.27	0	0	7.13	100	0	0	29.51
B_6	100	100	0	100	100	100	100	100
U_{108}	0	92.87	0	0	100	80.23	0.37	100
U_{117}	0	0	0	0	0	0	0	1.71
U_{170}	5.63	100	0	19.77	100	0	0	43.99
U_{196}	85.26	100	0	99.63	100	100	0	100
U_{216}	0	70.49	0	0	98.29	56.01	0	0

(d) Knowledge Transfer

$p_{\mathbf{3}}(\cdot,\cdot)$	B_7	B_8	U_{25}	U_{51}	U_{61}	U_{108}	U_{117}	U_{136}	U_{170}	U_{196}
B_7	0	0	5.83	0	0	0	0	0	6.99	36.92
B_8	100	0	100	97.27	0	52.58	0	52.26	100	100
U_{25}	94.17	0	0	0	0	0	0	0	89.29	89.29
U_{51}	100	2.73	100	0	0	0	0	10.75	100	100
U_{61}	100	100	100	100	0	100	100	100	100	100
U_{108}	100	47.42	100	100	0	0	25.01	52.24	100	100
U_{117}	100	100	100	100	0	74.99	0	68.77	100	100
U_{136}	100	47.74	100	89.25	0	47.76	31.23	0	100	100
U_{170}	93.01	0	10.71	0	0	0	0	0	0	89.29
U_{196}	63.08	0	10.71	0	0	0	0	0	10.71	0

Further information can be obtained looking at the pairwise winning indices in Tables 2.14(a)-2.14(d):

- At a comprehensive level, U_{61} is preferred to all but one the other universities in the first nondominated front with a frequency equal to the 100% while it is preferred to U_{136} with a frequency of the 93.29%. Analogously, U_{136} is preferred to all but two other universities in the first nondominated front with a frequency equal to the 100%. Indeed, it is preferred to U_{61} with a frequency of the 6.71% while, almost always, it is preferred to U_{129} ($p_0(U_{136}, U_{129}) = 99.76\%$). Looking at the worst universities in the first nondominated front, U_{35} could be considered as a bad university since it is never preferred to the most part of the universities in this front apart from U_{25}, $U64$ and U_{170} to which it is sometimes preferred with frequencies not very high;

- With respect to TL, universities belonging to block B_2, that are U_{117} and U_{216} are obviously always preferred to all other universities, while U_{196} is really bad since it is never preferred to any other university belonging to the first nondominated front;

- With respect to R, the universities belonging to block B_6, that are U_{61} and U_{136} are always preferred to all the other universities, while U_{117} could be considered the worst among the twelve universities at hand since it is only preferred to U_{216} with a frequency equal to the 1.71%;

- With respect to KT, U_{61} is preferred to all other universities since it gets the best performances on all elementary subcriteria descending from this macro-criterion, while U_{117} could be considered a quite well university with respect to KT since it is preferred to all other universities (apart from U_{61}) with a frequency at least equal to the 68.77%. Analogously, universities belonging to the block B_7, that are U_{35} and U_{64}, could be considered really bad since all other universities are almost always preferred to them.

Table 2.15: Barycenter values of the Möbius representation of compatible capacities

$m(\{MGR\})$	$m(\{MGOT\})$	$m(\{NRP\})$	$m(\{CR\})$	$m(\{PTCP\})$	$m(\{NPA\})$	$m(\{NSO\})$	$m(\{RKTR\})$	$m(\{MGR, MGOT\})$	$m(\{MGR, NRP\})$	$m(\{MGR, CR\})$	$m(\{MGR, PTCP\})$
0.0406	0.0841	0.0504	0.1119	0.1485	0.0636	0.1277	0.1646	0.0139	0.0077	0.0506	0.0586
$m(\{MGR, NPA\})$	$m(\{MGR, NSO\})$	$m(\{MGR, RKTR\})$	$m(\{MGOT, NRP\})$	$m(\{MGOT, CR\})$	$m(\{MGOT, PTCP\})$	$m(\{MGOT, NPA\})$	$m(\{MGOT, NSO\})$	$m(\{MGOT, RKTR\})$	$m(\{NRP, CR\})$	$m(\{NRP, PTCP\})$	$m(\{NRP, NPA\})$
0.0085	0.0005	0.0058	0.0238	-0.0027	-0.0249	0.0067	0.0103	0.0009	0.0479	0.0581	0.0117
$m(\{NRP, NSO\})$	$m(\{NRP, RKTR\})$	$m(\{CR, PTCP\})$	$m(\{CR, NPA\})$	$m(\{CR, NSO\})$	$m(\{CR, RKTR\})$	$m(\{PTCP, NPA\})$	$m(\{PTCP, NSO\})$	$m(\{PTCP, RKTR\})$	$m(\{NPA, NSO\})$	$m(\{NPA, RKTR\})$	$m(\{NSO, RKTR\})$
0.0100	0.0147	-0.0464	0.0205	0.0163	-0.0082	0.0113	-0.0053	0.0024	-0.0147	-0.0119	-0.0573

In order to get a ranking of the considered universities with respect to TL, R, KT and at the comprehensive level, we computed the barycenter of the Möbius representation of capacities compatible with the preferences provided by the DM. Their values are shown in Table 2.15. From this table one can conclude that, considered alone, the most important criterion is RKTR ($m(\{RKTR\}) = 0.1646$), followed by PTCP ($m(\{PTCP\}) = 0.1485$) and NSO ($m(\{NSO\}) = 0.1277$), while MGR is the least important one ($m(\{MGR\}) = 0.0406$). Moreover, apart from information provided by the DM about interactions about some elementary subcriteria, Table 2.15 shows other interactions, like positive interaction between MGR and MGOT or negative interaction between NSO and RKTR.

Computing the Choquet integral value for each university using the barycenter of the Möbius representations shown in Table 2.15, we get four complete rankings of universities at a comprehensive level and at the levels of macro-criteria. In Tables 2.16(a)-2.16(d) we show the complete rankings of the twelve universities in the first nondominated front underlying their positions in the full rankings, that are those obtained considering the 220 universities at hand.

One can observe that the Tech U Denmark is the best among the considered universities at the comprehensive level as well as on R and KT, while it fills the 5th position with respect to TL. It is interesting to note that university of Trieste has a high position with respect to R ($5th$) while it has a bad position with respect to KT ($35th$). Lomonosow Moscow State University behaves exactly in the opposite way, getting a bad position with respect to R ($25th$) and a good position with respect to KT ($6th$). These observations shed light on the usefulness of the MCHP in providing evaluable

Table 2.16: Complete rankings considering barycenter of the Möbius representations shown in Table 2.15

(a) Comprehensive level

Position in the complete ranking	University	Country
1*st*	Tech U Denmark	Denmark
2*nd*	Newcastle U	United Kingdom
31*th*	U Limerick	Ireland
32*th*	Mondragon U	Spain
41*th*	WHU School of Management	Germany
53*th*	Bocconi University	Italy
54*th*	U Trieste	Italy
58*th*	Lomonosow Moscow State U	Russia
67*th*	U Cordoba	Spain
78*th*	U Salamanca	Spain
91*th*	Dublin Inst. Tech	Ireland
105*th*	Budapest U Tech & Economics	Hungary

(b) Teaching and Learning

Position in the complete ranking	University	Country
1*st*	Lomonosow Moscow State U	Russia
	WHU School of Management	Germany
2*nd*	U Limerick	Ireland
	Mondragon U	Spain
	Newcastle U	United Kingdom
3*rd*	Bocconi University	Italy
	U Salamanca	Spain
4*th*	U Cordoba	Spain
5*th*	Tech U Denmark	Denmark
6*th*	Dublin Inst. Tech	Ireland
8*th*	Budapest U Tech & Economics	Hungary
12*th*	U Trieste	Italy

(c) Research

Position in the complete ranking	University	Country
1*st*	Tech U Denmark	Denmark
	Newcastle U	United Kingdom
5*th*	U Trieste	Italy
9*th*	Bocconi University	Italy
	Dublin Inst. Tech	Ireland
	Mondragon U	Spain
13*th*	U Limerick	Ireland
17*th*	WHU School of Management	Germany
18*th*	U Salamanca	Spain
19*th*	Budapest U Tech & Economics	Hungary
	U Cordoba	Spain
25*th*	Lomonosow Moscow State U	Russia

(d) Knowledge Transfer

Position in the complete ranking	University	Country
1*st*	Tech U Denmark	Denmark
6*th*	Lomonosow Moscow State U	Russia
9*th*	Mondragon U	Spain
	WHU School of Management	Germany
10*th*	U Limerick	Ireland
11*th*	Newcastle U	United Kingdom
16*th*	U Cordoba	Spain
28*th*	Bocconi University	Italy
31*th*	U Salamanca	Spain
35*th*	U Trieste	Italy
36*th*	Budapest U Tech & Economics	Hungary
	Dublin Inst. Tech	Ireland

insight into the problem at hand at different nodes of the hierarchy of criteria.

Even if we performed the analysis of the results for the alternatives belonging to the first nondominated front, for the sake of completeness, in Table 2.17 we list the first ten universities in the ranking at comprehensive level obtained considering the barycenter of the Möbius representation of the capacities compatible with the preference information provided by the DM. Moreover, we reported also the rank acceptability indices of the same universities with respect to the first five positions in the ranking.

Table 2.17: First ten universities in the ranking at comprehensive level obtained by considering the barycenter of the Möbius representations of the capacities compatible with the preferences provided by the DMs. Moreover, we provide the rank acceptability indices of the same universities for the first five positions.

Position	**University**	**Country**	b_k^1	b_k^2	b_k^3	b_k^4	b_k^5	b_k^6	b_k^7	b_k^8	b_k^9	b_k^{10}
1st	Tech U Denmark	Denmark	93.29	6.71	0	0	0	0	0	0	0	0
2nd	Newcastle U	United Kingdom	6.71	60.18	14.36	6.99	8.88	0.61	1.08	0.32	0.3	0.31
3th	Eindhoven U Tech	The Netherlands	0	33.11	16.54	34.88	10.72	4.75	0	0	0	0
4th	U Liverpool	United Kingdom	0	0	50.12	20.91	6.44	16.13	1.46	1.58	0.84	1.07
5th	U Bern	Switzerland	0	0	0	11	38.28	16.13	21	4.24	1.39	0.85
6th	Karlsruhe Inst. Tech (Kinst. Tech)	Denmark	0	0	7.49	12.43	17.79	26.64	12.56	18.71	3.92	0.41
7th	Tech U München	Germany	0	0	11.26	8.47	9.22	14.52	26.74	10.58	9.3	3.23
8th	U Liege	Belgium	0	0	0	0.19	0.51	1.29	3.91	8.51	16.29	21.56
9th	U Stuttgart	Germany	0	0	0	0.89	2.46	2.03	7.37	13.49	10.67	10.63
10th	U Groningen	The Netherlands	0	0	0	0	0	4.52	5.53	10.71	18.63	10.21

Looking at Table 2.17, one can argue that something is wrong in the presented results since only two of the universities in the first ten positions in the comprehensive ranking belong to the first nondominated front, that are the Tech U Denmark and the Newcastle U, even if it is not the case. Indeed, the fact that a university belong to the first nondominated front means only that there is not any other university dominating it in consequence of its excellence in one or more of the elementary subcriteria. This does not mean that at comprehensive level, that is considering all elementary subcriteria simultaneously, a university having not any excellence in some elementary subcriteria but having in average good performance could not be a good university. For example, the Liverpool university belongs to the 2nd nondominated front but its performances are such that it fills the 3rd position in the final ranking with a frequency of 50.12%. Even more, we could observe that three of the first ten universities belong to the 2nd nondominated front (Eindhoven U Tech and U Liverpool), three at the 3rd nondominated front (U Bern, Karlsruhe Inst. Tech and Tech U München), while two belong to the 4th nondominated front (U Stuttgart and U Groningen). Once more, we would like to underline that, even if we performed the analysis of the results for the universities belonging to the first nondominated front, the DM could make a similar analysis on every other subset of universities (s)he is interested in, therefore universities in another nondominated front or universities belonging to the same country and so on. The full list of the considered universites as well as the result obtained by applying the NAROR and the SMAA methodologies are available at data-MCHP-NAROR-SMAA.

2.2.4 Conclusions

In this paper, we presented a methodology of handling a hierarchical structure of interacting criteria in the multiple criteria ranking problem. To this end, we applied the Multiple Criteria Hierarchy Process with the Choquet integral preference model. The preference information provided by the user in the course of the decision aiding process has the form of pairwise comparisons of some alternatives and some criteria at different levels of the hierarchy of criteria. The set of instances of the Choquet integral compatible with this preference information is identified using the Robust Ordinal Regression (ROR). Then, Stochastic Multiobjective Acceptability Analysis (SMAA) is applied on this set of compatible instances, leading to recommendations in the form of complete rankings of alternatives at the comprehensive level of the hierarchy of criteria and with respect to all subcriteria excluding the elementary ones. SMAA provides, moreover, many useful indices permitting to assess the relative quality of particular alternatives in different nodes of the hierarchy tree, i.e., with respect to different macro-criteria..

The presented methodology performs a constructive learning process in which the user learns from the results supplied by ROR and SMAA indices, and the method learns from the preference information supplied incrementally by the user in successive iterations. This process ceases when the obtained recommendations and indices are conclusive enough for the user.

We envisage to apply the hierarchical Choquet integral preference model in conjunction with ROR and SMAA in case of criteria involving different evaluation scales. In this case, the method presented recently in [4] can be used to construct a common scale without the need of normalizing the evaluations.

Appendix

Proof of Proposition 2.2.1

We shall prove Proposition 2.2.1 by induction over α.

- First, let us prove the thesis for $\alpha = 1$. In this case, considering criterion $G_{(\mathbf{r},w_1)}$ as subcriterion of criterion $G_{\mathbf{r}}$ at the level k, we have

$$m^k_{\mathbf{r}}(\{G_{(\mathbf{r},w_1)}\}) = \mu^k_{\mathbf{r}}(\{G_{(\mathbf{r},w_1)}\}) = \frac{\mu(E(\{G_{(\mathbf{r},w_1)}\}))}{\mu(E(G_{\mathbf{r}}))} = \frac{\displaystyle\sum_{T\subseteq E(G_{(\mathbf{r},w_1)})} m(T)}{\mu(E(G_{\mathbf{r}}))}.$$

  The first equality is obtained by eq. (2.26) defining the Möbius transformation $m^k_{\mathbf{r}}$ of the capacity $\mu^k_{\mathbf{r}}$; the second equality is obtained by equation (2.22) defining the capacity $\mu^k_{\mathbf{r}}$ in terms of the capacity μ while the third one is obtained by equation (2.13) defining the Möbius transformation m of the capacity μ.

- Let us suppose that the thesis is true for $\alpha = n-1$, that is, for all $\left\{G_{(\mathbf{r},w_1)}, \ldots, G_{(\mathbf{r},w_{n-1})}\right\} \subseteq \mathcal{G}^k_{\mathbf{r}}$,

$$m^k_{\mathbf{r}}(\{G_{(\mathbf{r},w_1)}, \ldots, G_{(\mathbf{r},w_{n-1})}\}) = \frac{\displaystyle\sum_{\substack{T_1\subseteq E(G_{(\mathbf{r},w_1)}),\, T_1\neq\emptyset,\\ \ldots\\ T_\alpha\subseteq E(G_{(\mathbf{r},w_{n-1})}),\, T_{n-1}\neq\emptyset,}} m(\{T_1, \ldots, T_{n-1}\})}{\mu(E(G_{\mathbf{r}}))}.$$

- Now, let us prove that the thesis is true for $\alpha = n$.
  Let $\left\{G_{(\mathbf{r},w_1)}, \ldots, G_{(\mathbf{r},w_n)}\right\} \subseteq \mathcal{G}^k_{\mathbf{r}}$ and let us compute $\mu^k_{\mathbf{r}}\left(\left\{G_{(\mathbf{r},w_1)}, \ldots, G_{(\mathbf{r},w_n)}\right\}\right)$.

– By equation (2.26), we have that

$$\mu_{\mathbf{r}}^{k}\left(\left\{G_{(\mathbf{r},w_1)},\ldots,G_{(\mathbf{r},w_n)}\right\}\right)=\sum_{T\subseteq\left\{G_{(\mathbf{r},w_1)},\ldots,G_{(\mathbf{r},w_n)}\right\}} m_{\mathbf{r}}^{k}(T)=\sum_{T\subset\left\{G_{(\mathbf{r},w_1)},\ldots,G_{(\mathbf{r},w_n)}\right\}} m_{\mathbf{r}}^{k}(T)+$$

$$+m_{\mathbf{r}}^{k}\left(\left\{G_{(\mathbf{r},w_1)},\ldots,G_{(\mathbf{r},w_n)}\right\}\right)=\sum_{\beta=1}^{n} m_{\mathbf{r}}^{k}\left(\left\{G_{(\mathbf{r},w_\beta)}\right\}\right)+\sum_{\{\beta_1,\beta_2\}\subset\{w_1,\ldots,w_n\}} m_{\mathbf{r}}^{k}\left(\left\{G_{(\mathbf{r},\beta_1)},G_{(\mathbf{r},\beta_2)}\right\}\right)+$$

$$+\ldots+\sum_{\{\beta_1,\ldots,\beta_{n-1}\}\subset\{1,\ldots,n\}} m_{\mathbf{r}}^{k}\left(\left\{G_{(\mathbf{r},\beta_1)},\ldots,G_{(\mathbf{r},\beta_{n-1})}\right\}\right)+m_{\mathbf{r}}^{k}\left(\left\{G_{(\mathbf{r},w_1)},\ldots,G_{(\mathbf{r},w_n)}\right\}\right).$$

For the inductive hypothesis, we have therefore that

$$\mu_{\mathbf{r}}^{k}\left(\left\{G_{(\mathbf{r},w_1)},\ldots,G_{(\mathbf{r},w_n)}\right\}\right)=\sum_{\beta=1}^{n}\frac{\sum_{T_\beta\subseteq E\left(G_{(\mathbf{r},w_\beta)}\right)} m(T_\beta)}{\mu(E(G_{\mathbf{r}}))}+$$

$$+\sum_{\{\beta_1,\beta_2\}\subset\{w_1,\ldots,w_n\}}\frac{\sum_{\substack{T_{\beta_1}\subseteq E\left(G_{(\mathbf{r},w_{\beta_1})}\right),T_{\beta_1}\neq\emptyset,\\ T_{\beta_2}\subseteq E\left(G_{(\mathbf{r},w_{\beta_2})}\right),T_{\beta_2}\neq\emptyset}} m\left(\{T_{\beta_1},T_{\beta_2}\}\right)}{\mu(E(G_{\mathbf{r}}))}+\ldots+ \tag{2.31}$$

$$+\sum_{\{\beta_1,\ldots,\beta_{n-1}\}\subset\{1,\ldots,n\}}\frac{\sum_{\substack{T_{\beta_1}\subseteq E\left(G_{(\mathbf{r},w_{\beta_1})}\right),T_{\beta_1}\neq\emptyset,\\ \ldots\\ T_{\beta_{n-1}}\subseteq E\left(G_{(\mathbf{r},w_{\beta_{n-1}})}\right),T_{\beta_{n-1}}\neq\emptyset}} m\left(\left\{T_{\beta_1},\ldots,T_{\beta_{n-1}}\right\}\right)}{\mu(E(G_{\mathbf{r}}))}+m_{\mathbf{r}}^{k}(\left\{G_{(\mathbf{r},w_1)},\ldots,G_{(\mathbf{r},w_\alpha)}\right\}).$$

– From equation (2.22) we have that

$$\mu_{\mathbf{r}}^{k}\left(\left\{G_{(\mathbf{r},w_1)},\ldots,G_{(\mathbf{r},w_n)}\right\}\right)=\frac{\mu\left(E\left(\left\{G_{(\mathbf{r},w_1)},\ldots,G_{(\mathbf{r},w_n)}\right\}\right)\right)}{\mu(E(G_{\mathbf{r}}))}=\frac{\sum_{T\subseteq E\left(\left\{G_{(\mathbf{r},w_1)},\ldots,G_{(\mathbf{r},w_n)}\right\}\right)} m(T)}{\mu(E(G_{\mathbf{r}}))}=$$

$$=\sum_{\beta=1}^{n}\frac{\sum_{T_\beta\subseteq E\left(G_{(\mathbf{r},w_\beta)}\right)} m(T_\beta)}{\mu(E(G_{\mathbf{r}}))}+\sum_{\{\beta_1,\beta_2\}\subset\{w_1,\ldots,w_\alpha\}}\frac{\sum_{\substack{T_{\beta_1}\subseteq E\left(G_{(\mathbf{r},w_{\beta_1})}\right),\,T_{\beta_1}\neq\emptyset,\\ T_{\beta_2}\subseteq E\left(G_{(\mathbf{r},w_{\beta_2})}\right),\,T_{\beta_2}\neq\emptyset}} m\left(\{T_{\beta_1},T_{\beta_2}\}\right)}{\mu(E(G_{\mathbf{r}}))}+$$

$$+\ldots+\sum_{\{\beta_1,\ldots,\beta_{n-1}\}\subset\{1,\ldots,n\}}\frac{\displaystyle\sum_{\substack{T_{\beta_1}\subseteq E\left(G_{(\mathbf{r},w_{\beta_1})}\right),\,T_{\beta_1}\neq\emptyset,\\ \ldots\\ T_{\beta_{\alpha-1}}\subseteq E\left(G_{(\mathbf{r},w_{\beta_{n-1}})}\right),\,T_{\beta_{n-1}}\neq\emptyset}} m\left(\left\{T_{\beta_1},\ldots,T_{\beta_{n-1}}\right\}\right)}{\mu(E(G_{\mathbf{r}}))}+ \tag{2.32}$$

$$+\frac{\displaystyle\sum_{\substack{T_1\subseteq E\left(G_{(\mathbf{r},w_1)}\right),\,T_1\neq\emptyset,\\ \ldots\\ T_n\subseteq E(G_{(\mathbf{r},w_n)}),\,T_n\neq\emptyset}} m\left(\{T_1\cup\ldots\cup T_n\}\right)}{\mu(E(G_{\mathbf{r}}))}$$

From equations (2.31) and (2.32), we get the thesis.

Proof of Proposition 2.2.2

Let m the Möbius representation of a q-additive capacity μ, $\left\{G_{(\mathbf{r},w_1)},\ldots,G_{(\mathbf{r},w_\alpha)}\right\}\subseteq\mathcal{G}_{\mathbf{r}}^k$ with $\alpha>q$ and $m_{\mathbf{r}}^k$ the Möbius representation of the capacity $\mu_{\mathbf{r}}^k$. By Proposition (2.2.1), we have that

$$m_{\mathbf{r}}^k\left(\left\{G_{(\mathbf{r},w_1)},\ldots,G_{(\mathbf{r},w_\alpha)}\right\}\right)=\frac{\displaystyle\sum_{\substack{T_1\subseteq E\left(G_{(\mathbf{r},w_1)}\right),\,T_1\neq\emptyset,\\ \ldots\\ T_\alpha\subseteq E\left(G_{(\mathbf{r},w_\alpha)}\right),\,T_\alpha\neq\emptyset}} m\left(\{T_1,\ldots,T_\alpha\}\right)}{\mu(E(G_{\mathbf{r}}))}.$$

Observing that the set $\{T_1,\ldots,T_\alpha\}$ will contain at least $q+1$ elements (since $\alpha>q$) and that the capacity μ is q-additive, we get that $m_{\mathbf{r}}^k\left(\left\{G_{(\mathbf{r},w_1)},\ldots,G_{(\mathbf{r},w_\alpha)}\right\}\right)=0$ for all $\left\{G_{(\mathbf{r},w_1)},\ldots,G_{(\mathbf{r},w_\alpha)}\right\}\subseteq\mathcal{G}_{\mathbf{r}}^k$ with $\alpha>q$.

Proof of Proposition 2.2.3

1. Given $G_{(\mathbf{r},w)}\in\mathcal{G}_{\mathbf{r}}^k$, by equations (2.27) and (2.2.1) and, considering that the capacity μ is 2-additive, we have that

$$\varphi_{\mathbf{r}}^k\left(\left\{G_{(\mathbf{r},w)}\right\}\right)=\sum_{\mathcal{F}\subseteq\mathcal{G}_{\mathbf{r}}^k:\,G_{(\mathbf{r},w)}\in\mathcal{F}}\frac{m_{\mathbf{r}}^k(\mathcal{F})}{|\mathcal{F}|}=m_{\mathbf{r}}^k\left(\left\{G_{(\mathbf{r},w)}\right\}\right)+\sum_{G_{(\mathbf{r},w_1)}\subseteq\mathcal{G}_{\mathbf{r}}^k\setminus\left\{G_{(\mathbf{r},w)}\right\}}\frac{m_{\mathbf{r}}^k\left(\left\{G_{(\mathbf{r},w)},G_{(\mathbf{r},w_1)}\right\}\right)}{2}=$$

$$=\sum_{T\subseteq E\left(G_{(\mathbf{r},w)}\right)}\frac{m(T)}{\mu(E(G_{\mathbf{r}}))}+\frac{1}{2}\sum_{G_{(\mathbf{r},w_1)}\subseteq\mathcal{G}_{\mathbf{r}}^k\setminus\left\{G_{(\mathbf{r},w)}\right\}}\frac{\displaystyle\sum_{\substack{T_1\subseteq E\left(G_{(\mathbf{r},w)}\right),\,T_1\neq\emptyset,\\ T_2\subseteq E\left(G_{(\mathbf{r},w_1)}\right),\,T_2\neq\emptyset}} m(\{T_1,T_2\})}{\mu(E(G_{\mathbf{r}}))}=$$

$$= \frac{1}{\mu(E(G_{\mathbf{r}}))} \left[\sum_{\mathbf{t} \in E\left(G_{(\mathbf{r},w)}\right)} m(g_{\mathbf{t}}) + \sum_{\mathbf{t}_1, \mathbf{t}_2 \in E\left(G_{(\mathbf{r},w)}\right)} m\left(\{g_{\mathbf{t}_1}, g_{\mathbf{t}_2}\}\right) \right] + \frac{1}{\mu(E(G_{\mathbf{r}}))} \sum_{\substack{\mathbf{t}_1 \in E\left(G_{(\mathbf{r},w)}\right), \\ \mathbf{t}_2 \in E\left(\mathcal{G}_{\mathbf{r}}^k \setminus G_{(\mathbf{r},w)}\right)}} \frac{m\left(\{g_{\mathbf{t}_1}, g_{\mathbf{t}_2}\}\right)}{2} =$$

$$= \left[\sum_{\mathbf{t} \in E\left(G_{(\mathbf{r},w)}\right)} m(g_{\mathbf{t}}) + \sum_{\mathbf{t}_1, \mathbf{t}_2 \in E\left(G_{(\mathbf{r},w)}\right)} m\left(\{g_{\mathbf{t}_1}, g_{\mathbf{t}_2}\}\right) + \sum_{\substack{\mathbf{t}_1 \in E\left(G_{(\mathbf{r},w)}\right), \\ \mathbf{t}_2 \in E\left(\mathcal{G}_{\mathbf{r}}^k \setminus G_{(\mathbf{r},w)}\right)}} \frac{m\left(\{g_{\mathbf{t}_1}, g_{\mathbf{t}_2}\}\right)}{2} \right] \frac{1}{\mu\left(E(G_{\mathbf{r}})\right)}.$$

2. Given $G_{(\mathbf{r},w_1)}, G_{(\mathbf{r},w_2)} \in \mathcal{G}_{\mathbf{r}}^k$, by equation (2.28) and Proposition 2.2.1 and, considering that the capacity μ is 2-additive, we have that

$$\varphi_{\mathbf{r}}^k \left(\left\{G_{(\mathbf{r},w_1)}, G_{(\mathbf{r},w_2)}\right\}\right) = \sum_{\mathcal{F} \subseteq \mathcal{G}_{\mathbf{r}}^k \colon G_{(\mathbf{r},w_1)}, G_{(\mathbf{r},w_2)} \in \mathcal{F}} \frac{m_{\mathbf{r}}^k(\mathcal{F})}{|\mathcal{F}| - 1} = m_{\mathbf{r}}^k \left(\left\{G_{(\mathbf{r},w_1)}, G_{(\mathbf{r},w_2)}\right\}\right) =$$

$$= \sum_{\substack{T_1 \subseteq E\left(G_{(\mathbf{r},w_1)}\right),\, T_1 \neq \emptyset, \\ T_2 \subseteq E\left(G_{(\mathbf{r},w_2)}\right),\, T_2 \neq \emptyset}} \frac{m\left(\{T_1, T_2\}\right)}{\mu(E(G_{\mathbf{r}}))} = \left[\sum_{\substack{\mathbf{t}_1 \in E(G_{(\mathbf{r},w_1)}), \\ \mathbf{t}_2 \in E(G_{(\mathbf{r},w_2)})}} m\left(\{g_{\mathbf{t}_1}, g_{\mathbf{t}_2}\}\right) \right] \frac{1}{\mu(E(G_{\mathbf{r}}))}$$

Chapter 3

Contributions related to the Interaction between criteria

In section 1.1 we stated that, if the mutual preference independence is satisfied, then the preferences provided by the DM can be represented by an additive value function. But, in general, the set of criteria is not always mutually preferentially independent. For this reason, in the last decades, the non-additive integrals and, in particular, the Choquet integral preference model began to be used very frequently in literature. Two drawbacks of the Choquet integral preference model have been highlighted, that are, the great number of parameters necessary to its application ($2^n - 2$ in case of n criteria) and the requirement that all the evaluations of the alternatives on different criteria are given on a common scale. While the first problem has been dealt by using ordinal regression [6, 119] and non-additive robust ordinal regression [8], in the first two contributions of this chapter we dealt with the problem of the construction of a common scale.

In the first contribution, we proposed a procedure to build a common scale for the evaluations of the alternatives and, since as in the case of the models compatible with the preferences provided by the DM, more than one common scale could be built, we applied the SMAA methodology to take into account the plurality of common scales that can be built.

In the second contribution, instead, we applied AHP to build a common scale proposing a new methodology to reduce the number of pairwise comparisons requested in AHP.

In the third contribution we proposed to apply the Choquet integral preference model to the evolutionary multiobjective optimization. The contribution proposed for the first time the application of the Choquet integral preference model to the evolutionary multiobjective field and the Choquet integral is used to address the research on the region of the Pareto front most interesting for the

DM. The proposed method, called NEMO-II-Ch, has been tested on several benchmark problems on different dimensions outperforming in most of the cases the other methods with which it has been compared.

The three contributions related to the interaction between criteria are provided in sections 3.1, 3.2 and 3.3, respectively.

3.1 Stochastic Multiobjective Acceptability Analysis for the Choquet integral preference model and the scale construction problem

3.1.1 Introduction

In Multiple Criteria Decision Aiding (MCDA) (see [51] for a collection of surveys on MCDA), an alternative a_k, belonging to a finite set of l alternatives $A = \{a_1, \dots, a_l\}$, is evaluated on the basis of a family of n criteria $G = \{g_1, \dots, g_n\}$. For example, in a car decision problem, the set A is composed of different car models while criteria in G are features of the cars taken into consideration, such as, price, maximum speed, acceleration and so on. In the description of the methodology we are proposing, we shall suppose, for the sake of simplicity, that $g_i : A \rightarrow X_i \subseteq \mathbb{R}$, which does not exclude X_i from being a number-coded ordinal scale.

To give a recommendation for the decision making problem at hand, the evaluations of the alternatives on all criteria have to be aggregated. In literature, the three main aggregation approaches are the Multi-Attribute Value Theory (MAVT) [109], the outranking methods [141] (among which the most well known are the ELECTRE [54] and PROMETHEE [27] methods) and the Dominance-based Rough Set Approach (DRSA, see [82]). In the following, we shall describe MAVT being the aggregation approach used in the paper.

MAVT takes into consideration an overall value function $U\colon \mathbb{R}^n \rightarrow \mathbb{R}$ with $U(g_1(a_k), \dots, g_n(a_k)) = U(a_k)$, such that alternative a_k is indifferent to alternative a_h iff $U(a_k) = U(a_h)$ and a_k is preferred to a_h iff $U(a_k) > U(a_h)$ for any a_k, $a_h \in A$. The value functions used in MAVT can take different forms, but, the most common is the additive one. It is based on the preference independence of the criteria [109, 174], even if it is an unrealistic assumption or a too strong simplification, since in many cases the criteria can be interacting. For instance, let us consider the car decision problem introduced above. On one hand, maximum speed and acceleration are redundant criteria because, in general, speedy cars also have a good acceleration. Therefore, even if these two criteria can be very important, their comprehensive importance is smaller than the sum of the importance of the two criteria considered separately. On the other hand, maximum speed and price lead to a synergy effect, because a speed car having also a low price is very well appreciated. For such a reason, the comprehensive importance of these two criteria should be greater than the sum of the importance of the two criteria considered separately.

In the MAVT context, multiplicative and multilinear value functions are able to take into account interactions between criteria but, due to the high number of parameters that have to be elicited from the DM, its use results of marginal relevance in real world applications [161]. Recently, interactions between criteria have been considered also in ELECTRE methods [52] and PROMETHEE methods [34].
Within MCDA, the interaction between criteria has frequently been dealt by using non-additive integrals, the most well known of which are the Choquet integral [31] and the Sugeno integral [162] (see [68, 73, 74] for a comprehensive survey on the use of non-additive integrals in MCDA; see also [71, 72, 80, 90] for some recently proposed extensions of non-additive integrals useful in MCDA).

The two main drawbacks of the Choquet integral preference model are the great number of parameters that have to be elicited in order to apply it and the requirement that criteria are on a common scale.
Regarding the elicitation of the preference parameters, the DM can provide direct or indirect preference information [6, 119]. The DM gives direct preference information when she provides directly all the values of the parameters present in the model. The DM supplies indirect preference information (see e.g. [97]) when she provides some preferences between alternatives or comparisons about importance and interaction of criteria from which compatible preference parameters can be inferred. With respect to the Choquet integral preference model, the inference of the preference parameters is really challenging, but several methodologies have been proposed in literature [70, 119].

Concerning the common scale problem, let us mention that the Choquet integral preference model requires that evaluations on different criteria have to be compared between them. For example, in the considered car decision problem, the DM should be able to compare the speed of a car with its acceleration estimating, for example, if the maximum speed of 200 km/h is as valuable as a price of $35,000$ €. This problem is quite well known in literature (see e.g. [124]) but, to the best of our knowledge, very few contributions tackled the problem (e.g. [6] proposes a search of a common scale through Monte Carlo simulation). In this paper, we shall deal with these two drawbacks of the Choquet integral preference model.

The elicitation of the preference parameters has been already taken into account in our previous work [3], where the SMAA-Choquet methodology has been presented. In that paper, we have applied the Stochastic Multiobjective Acceptability Analysis (SMAA) (for a survey on SMAA methods see [164]) to explore the whole space of parameters compatible with some preference information provided by the DM related to the importance and the interaction of criteria.

The contributions of this paper are threefold:

1. SMAA-Choquet has been extended by taking into account also the DM's preference information regarding the pairwise comparison of some reference alternatives,

2. SMAA-Choquet has been also enhanced by including the possibility that the evaluations on criteria may be given imprecisely, that is the evaluation of each alternative on the considered criteria is not given punctually but as interval of possible evaluations,

3. SMAA-Choquet includes a procedure aiming to obtain a common scale for all considered criteria permitting, therefore, to apply the Choquet integral preference model.

The paper is organized as follows. In Section 2, we introduce the Choquet integral preference model, presented by a didactic example. In Section 3, we briefly describe the SMAA methods. Our simulation based approach, proposed in the context of the Choquet integral preference model, is introduced in Section 4 and illustrated by two examples in Section 5. Some conclusions and future directions of research are presented in Section 6.

3.1.2 The Choquet integral preference model

Very often the aggregation of the evaluations of an alternative on the considered criteria is done by means of the simplest additive value function, i.e. the *weighted sum*. It is obtained considering a vector of non-negative weights $\mathbf{w} = [w_1, ..., w_n]$ (one for each criterion in G), that permits to assign a value $U(a_k) = w_1 g_1(a_k) + \ldots + w_n g_n(a_k)$ to the alternative $a_k \in A$. Notice that, in the rest of the paper, we shall use the terms criterion g_i and criterion i interchangeably.

The weighted sum has several limitations to represent preferences (see e.g. [68, 109]) as illustrated by the following didactic example inspired by [73].

Example The dean of a technical school wants to evaluate students s_1, s_2 and s_3 whose marks on Mathematics and Physics are shown in Table 3.1.

Table 3.1: Students' evaluations on Mathematics and Physics given on a $[0, 30]$ scale

	Math	Phy
s_1	26	30
s_2	28	28
s_3	30	26

Since students good in Mathematics are in general good also in Physics, if there is a good mark in one of the two subjects, one can expect a good mark also in the other subject. Consequently, a student good in Mathematics and in Physics is of course appreciated, but the dean does not want to overvalue students having good marks in both subjects. Thus, for the dean, students s_1 and s_3 are preferred to student s_2.

In this case, we can say that there is a negative interaction (redundancy) between Mathematics and Physics. To represent the dean's preferences by means of the weighted sum model, the following inequalities should be satisfied:

$$\begin{aligned} w_{\text{Math}} \cdot 26 + w_{\text{Phy}} \cdot 30 &> w_{\text{Math}} \cdot 28 + w_{\text{Phy}} \cdot 28, \\ w_{\text{Math}} \cdot 30 + w_{\text{Phy}} \cdot 26 &> w_{\text{Math}} \cdot 28 + w_{\text{Phy}} \cdot 28, \end{aligned}$$

where w_{Math} and w_{Phy} are the weights of Mathematics and Physics, respectively. It is easily verified that the above inequalities are contradictory since:

$$w_{\text{Math}} \cdot (-2) + w_{\text{Phy}} \cdot 2 > 0 > w_{\text{Math}} \cdot (-2) + w_{\text{Phy}} \cdot 2$$

Thus, we have to conclude that, due to the redundancy between Mathematics and Physics, the weighted sum is not able to represent the dean's preferences. $\square$

In order to represent preferences in case of interaction between criteria, one has to use some preference model more general than the weighted sum. This is the case of the non-additive integrals among which the most well-known is the Choquet integral [31]. It proposes an extension of the weighted sum model to the case of interacting criteria and it is based on the concept of capacity (fuzzy measure) that assigns a weight to each subset of criteria. More precisely, denoting by 2^G the power set of G (i.e. the set of all subsets of G), the function $\mu : 2^G \rightarrow [0,1]$ is called a capacity (fuzzy measure) on 2^G if the following properties are satisfied:

1a) $\mu(\emptyset) = 0$ and $\mu(G) = 1$ (boundary conditions),

2a) $\forall\, S \subseteq T \subseteq G,\ \ \mu(S) \leq \mu(T)$ (monotonicity condition).

Intuitively, for all $T \subseteq G$, $\mu(T)$ can be interpreted as the comprehensive importance of the criteria from T considered as a whole.

Example (Continuation). To represent the importance of Mathematics and Physics taken singularly and considered together, one can set $\mu_1(\{Math\}) = \mu_1(\{Phy\}) = 0.6$ and $\mu_1(\{Math, Phy\}) = 1$. The difference $\mu_1(\{Math, Phy\}) - \mu_1(\{Math\}) - \mu_1(\{Phy\}) = -0.2$ represents the negative interaction between Mathematics and Physics since it is the difference between the importance of Mathematics and Physics considered as a whole ($\mu_1(\{Math, Phy\})$), and the sum of their importance when they are considered singularly ($\mu_1(\{Math\}) + \mu_1(\{Phy\})$). $\square$

If there is no interaction between the considered criteria, we have $\mu(S \cup T) = \mu(S) + \mu(T)$, for any $S, T \subseteq G$ such that $S \cap T = \emptyset$ and the capacity is called *additive*. If a capacity is additive then $\mu(T) = \sum_{i \in T} \mu(\{i\})$ and, consequently, the values $\mu(\{1\}), \mu(\{2\}) \ldots, \mu(\{n\})$ (corresponding to the weights w_i of the weighted sum model), are sufficient to rebuild the whole capacity μ.
Whenever the capacity is non-additive, in general, one has to assess $2^{|G|} - 2$ values $\mu(T), \emptyset \subset T \subset G$, since the values $\mu(\emptyset) = 0$ and $\mu(G) = 1$ are already known.

If the criteria from G are interacting and their importance is represented by a capacity μ, the weighted sum can be extended through the Choquet integral [31] that assigns the following value to each $a_k \in A$:

$$C_\mu(a_k) = \sum_{i=1}^{n} \left[g_{(i)}(a_k) - g_{(i-1)}(a_k) \right] \mu\left(N_i\right),$$

where $_{(\cdot)}$ stands for a permutation of the indices of criteria such that $g_{(1)}(a_k) \leq \ldots \leq g_{(n)}\left(a_k\right)$, $N_i = \{(i), \ldots, (n)\}$ and $g_{(0)} = 0$.

A meaningful and useful reformulation of the capacity μ and of the Choquet integral can be obtained by means of the Möbius representation of the capacity μ which is a function $m : 2^G \rightarrow \mathbb{R}$ [153] defined as follows:

$$\mu(S) = \sum_{T \subseteq S} m(T).$$

Note that if S is a singleton, i.e. $S = \{i\}$ with $i = 1, 2, \ldots, n$, then $\mu(\{i\}) = m(\{i\})$ while, if S is a couple (non-ordered pair) of criteria, i.e. $S = \{i, j\}$, then $\mu(\{i, j\}) = m(\{i\}) + m(\{j\}) + m(\{i, j\})$.
The Möbius representation $m(S)$ can be obtained from $\mu(S)$ as follows:

$$m(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \mu(T).$$

In terms of Möbius representation, properties **1a)** and **2a)** are, respectively, restated as:

1b) $m(\emptyset) = 0, \ \sum_{T \subseteq G} m(T) = 1,$

2b) $\forall\, i \in G$ and $\forall R \subseteq G \setminus \{i\}, \ m(\{i\}) + \sum_{T \subseteq R} m(T \cup \{i\}) \geq 0.$

The Choquet integral may be reformulated in terms of Möbius representation as follows:

$$C_\mu(a_k) = \sum_{T \subseteq G} m(T) \min_{i \in T} g_i\left(a_k\right). \tag{3.1}$$

Example (Continuation). The value assigned to student s_1 by the Choquet integral in terms of the capacity μ_1 is the following:

$$C_{\mu_1}(s_1) = g_{Math}(s_1) \cdot \mu_1(\{Math, Phy\}) + (g_{Phy}(s_1) - g_{Math}(s_1)) \cdot \mu_1(\{Phy\}) = 28.4.$$

This value can be explained as follows. The mark $g_{Math}(s_1) = 26$ is attained on both subjects and, therefore, it is multiplied by $\mu_1(\{Math, Phy\})$, i.e. the weight assigned to Mathematics and Physics considered as a whole. The mark $g_{Phy}(s_1) = 28$ is attained on Physics only and, consequently, the difference $g_{Phy}(s_1) - g_{Math}(s_1)$ is multiplied by $\mu_1(\{Phy\})$, i.e. the weight assigned to Physics considered singularly. Analogously, we get $C_{\mu_1}(s_2) = 28$ and $C_{\mu_1}(s_3) = 28.4$, so that $C_{\mu_1}(s_1) > C_{\mu_1}(s_2)$ and $C_{\mu_1}(s_3) > C_{\mu_1}(s_2)$. Therefore, we can conclude that the Choquet integral is able to represent the dean's preferences.

Observe also that the Möbius representation m_1 of the capacity μ_1 gives $m_1(\{Math\}) = m_1(\{Phy\}) = 0.6$ and $m_1(\{Math, Phy\}) = -0.2$ and, consequently, the Choquet integral related to student s_1 can be reformulated as follows in terms of the Möbius representation m_1:

$$C_{\mu_1}(s_1) = g_{Math}(s_1) \cdot m_1(\{Math\}) + g_{Phy}(s_1) \cdot m_1(\{Phy\}) + \min(g_{Math}(s_1), g_{Phy}(s_1)) \cdot m_1(\{Math, Phy\}) = 28.4$$

Considering its formulation in terms of Möbius representation, the Choquet integral can be explained as follows. The marks in Mathematics and Physics are multiplied by $m_1(\{Math\})$ and $m_1(\{Phy\})$ representing, in some form, the weights related to their additive components. However, the value so obtained has to be corrected by adding $\min(g_{Math}(s_1), g_{Phy}(s_1)) \cdot m_1(\{Math, Phy\})$ representing the negative interaction between Mathematics and Physics. The Choquet integral related to students s_2 and s_3 can be analogously reformulated in terms of the Möbius representation m_1. $\square$

With the aim of reducing the number of parameters to be elicited, in [69] the concept of k-additive capacity has been introduced. A capacity is called k-additive if $m(T) = 0$ for $T \subseteq G$ such that $|T| > k$.

Within an MCDA context, it is easier and more straightforward to consider 2-additive capacities since, in such case, the DMs have to express a preference information on positive and negative interactions between two criteria, neglecting more complex interactions among three, four and generally $k \leq n$ criteria. Moreover, by considering 2-additive measures the computational issue of determining the parameters is weakened, since only $n + \binom{n}{2}$ coefficients have to be assessed; specifically, in terms of Möbius representation, a value $m(\{i\})$ for every criterion i and a value $m(\{i, j\})$ for every couple of criteria $\{i,j\}$. For all these reasons, in the following we shall consider 2-additive capacities only. However, the methodology we are presenting can be applied to any capacity.

The value that a 2-additive capacity μ assigns to a set $S \subseteq G$ can be expressed in terms of the Möbius representation as follows:

$$\mu(S) = \sum_{i \in S} m(\{i\}) + \sum_{\{i,j\} \subseteq S} m(\{i,j\}), \ \forall S \subseteq G.$$

With regard to 2-additive capacities, properties **1b)** and **2b)** have, respectively, the following expressions:

1c) $m(\emptyset) = 0, \ \sum_{i \in G} m(\{i\}) + \sum_{\{i,j\} \subseteq G} m(\{i,j\}) = 1,$

2c) $\begin{cases} m(\{i\}) \geq 0, \ \forall i \in G, \\ m(\{i\}) + \sum_{j \in T} m(\{i,j\}) \geq 0, \ \forall i \in G \text{ and } \forall \, T \subseteq G \setminus \{i\}, \, T \neq \emptyset. \end{cases}$

In this case, the Choquet integral assigns to $a_k \in A$ the following value:

$$C_\mu(a_k) = \sum_{i \in G} m(\{i\}) \, g_i(a_k) + \sum_{\{i,j\} \subseteq G} m(\{i,j\}) \min\{g_i(a_k), g_j(a_k)\}. \tag{3.2}$$

Since, in this context, the importance of a criterion does not depend only on its importance as a single but also on its contribution to each coalition of criteria to which it participates, we recall the definitions of the importance of a criterion and of the interaction index for a couple of criteria. Taking into account the Möbius representation of a 2-additive capacity μ, the importance of criterion $i \in G$, expressed by the Shapley value [154], can be written as follows:

$$\varphi(\{i\}) = m(\{i\}) + \sum_{j \in G \setminus \{i\}} \frac{m(\{i,j\})}{2}.$$

The interaction index, expressing the sign and the magnitude of the interaction in a couple of criteria $\{i,j\} \subseteq G$ in case of a 2-additive Möbius representation of a capacity μ, is given by:

$$\varphi(\{i,j\}) = m(\{i,j\}).$$

Example (Continuation). Capacity μ_1 is trivially 2-additive and we have

$$\varphi(\{Math\}) = m_1(\{Math\}) + \frac{m_1(\{Math, Phy\})}{2} = 0.5.$$

Observing that $\varphi(\{Phy\}) = 0.5$ and, consequently, $\varphi(\{Math\}) = \varphi(\{Phy\})$, we can conclude that the marks in Mathematics and Physics have the same importance. Moreover, the value $\varphi(\{Math, Phy\}) = -0.2$ confirms that the two considered criteria are negatively interacting. □

3.1.3 SMAA

Stochastic Multiobjective Acceptability Analysis (SMAA) [113, 115] is a family of MCDA methods aiming to get recommendations on the problem at hand taking into account uncertainty or imprecision on the considered data and preference parameters. Several SMAA methods have been developed to deal with different MCDA problems: SMAA-2 has been presented in [115] for ranking problems, SMAA-O [114] has been introduced for multicriteria problems with ordinal criteria and SMAA-TRI [165] for sorting problems. Other two recent contributions related to SMAA and ROR have been presented in [105] and [106]. In the following, we shall describe SMAA-2 since, in this paper, we have considered ranking problems only.

In SMAA-2, the most commonly used value function is the linear one:

$$u(a_k, w) = \sum_{i=1}^{n} w_i g_i(a_k).$$

In order to take into account imprecision or uncertainty, SMAA-2 considers two probability distributions $f_W(w)$ and $f_\chi(\xi)$ on W and χ, respectively, where $W = \{(w_1, \ldots, w_n) \in \mathbb{R}^n \colon w_i \geq 0 \text{ and } \sum_{i=1}^{n} w_i = 1\}$ and χ is the evaluation space.

First of all, SMAA-2 introduces a ranking function relative to the alternative a_k:

$$rank(k, \xi, w) = 1 + \sum_{h \neq k} \rho\left(u(\xi_h, w) > u(\xi_k, w)\right),$$

where $\rho(false) = 0$ and $\rho(true) = 1$.

Then, for each alternative a_k, for each evaluation of alternatives $\xi \in \chi$ and for each rank $r = 1, \ldots, l$, SMAA-2 computes the set of weights of criteria for which alternative a_k assumes rank r:

$$W_k^r(\xi) = \{w \in W : rank(k, \xi, w) = r\}.$$

SMAA-2 is based on the computation of the following indices:

- *The rank acceptability index* measures the variety of different parameters compatible with the DM's preference information giving to the alternative a_k the rank r:

$$b_k^r = \int_{\xi \in \chi} f_\chi(\xi) \int_{w \in W_k^r(\xi)} f_W(w)\ dw\ d\xi;$$

  b_k^r gives the probability that alternative a_k has rank k and it is within the range $[0, 1]$.

- *The central weight vector* describes the preferences of a typical DM giving to a_k the best position and it is defined as follows:

$$w_k^c = \frac{1}{b_k^1} \int_{\xi \in \chi} f_\chi(\xi) \int_{w \in W^1(\xi)} f_W(w) w\ dw\ d\xi;$$

- *The confidence factor* is defined as the frequency of an alternative to be the preferred one with the preferences expressed by its central weight vector and it is given by:

$$p_k^c = \int_{\substack{\xi \in \chi : u(\xi_k, w_k^c) \geq u(\xi_h, w_k^c) \\ \forall h = 1, \ldots, l}} f_\chi(\xi)\ d\xi.$$

In the following, we shall consider also the frequency that an alternative a_h is weakly preferred to an alternative a_k in the space of the preference parameters (weight vectors in case of SMAA-2):

$$p_{hk} = \int_{w \in W} f_W(w) \int_{\xi \in \chi : u(\xi_h, w) \geq u(\xi_k, w)} f_\chi(\xi) d\xi\ dw.$$

Let us notice that the previous index p_{hk} is also known as pairwise winning index and it has been introduced in [116, 166].

From a computational point of view, the multidimensional integrals defining the considered indices are estimated by using the Monte Carlo method.

3.1.4 An extension of the SMAA method to the Choquet integral preference model

In this section, we shall present the SMAA-Choquet method putting together the Choquet integral preference model and the SMAA methodology.

As observed in Section 3.1.2, the use of the Choquet integral in terms of Möbius representation with a 2-additive capacity requires the evaluation of $n + \binom{n}{2}$ parameters and in order to assess these parameters, the DM is asked to provide some preference information in a direct or an indirect way. Generally, the indirect preference information requires less cognitive effort from the DM, and for this reason it is widely used in MCDA (see for example [8, 82, 97, 98]). In the following, we shall suppose that the DM is able to provide some indirect preference information and we shall use the 2-additive Choquet integral preference model expressed in terms of the Möbius representation.

Using the Choquet integral preference model, the DM can provide the following preference information:

- Comparisons related to importance and interaction of criteria:
  - criterion i is at least as important as criterion j (and we shall write $i \succsim j$): $\varphi(\{i\}) \geq \varphi(\{j\})$;
  - criterion i is more important than criterion j ($i \succ j$): $\varphi(\{i\}) > \varphi(\{j\})$;
  - criteria i and j have the same importance ($i \sim j$): $\varphi(\{i\}) = \varphi(\{j\})$;
  - criteria i and j are synergic: $\varphi(\{i, j\}) > 0$;
  - criteria i and j are redundant: $\varphi(\{i, j\}) < 0$.

- Comparisons between couples or quadruples of alternatives:
  - alternative a_k is at least as good as alternative a_h ($a_k \succsim a_h$): $C_\mu(a_k) \geq C_\mu(a_h)$;
  - alternative a_k is preferred to alternative a_h ($a_k \succ a_h$): $C_\mu(a_k) > C_\mu(a_h)$;
  - alternative a_k and a_h are indifferent ($a_k \sim a_h$): $C_\mu(a_k) = C_\mu(a_h)$;
  - alternative a_k is preferred to alternative a_h more than alternative a_s is preferred to alternative a_t ($(a_k, a_h) \succ^* (a_s, a_t)$): $C_\mu(a_k) - C_\mu(a_h) > C_\mu(a_s) - C_\mu(a_t)$;
  - the difference of preference between a_k and a_h is the same of the difference of preference between a_s and a_t ($(a_k, a_h) \sim^* (a_s, a_t)$): $C_\mu(a_k) - C_\mu(a_h) = C_\mu(a_s) - C_\mu(a_t)$.

Hereafter, we distinguish three sets of constraints:

- Monotonicity and boundary constraints,

$$\left.\begin{array}{l} m\left(\{\emptyset\}\right)=0,\ \sum\limits_{i\in G} m\left(\{i\}\right)+\sum\limits_{\{i,j\}\subseteq G} m\left(\{i,j\}\right)=1,\\ m\left(\{i\}\right)\geq 0,\ \forall i\in G,\\ m\left(\{i\}\right)+\sum\limits_{j\in T} m\left(\{i,j\}\right)\geq 0, \forall i\in G \text{ and } \forall\, T\subseteq G\setminus\{i\}\,, T\neq\emptyset, \end{array}\right\}(E^{MB})$$

- Constraints related to importance and interaction of criteria,

$$\left.\begin{array}{l} \varphi(\{i\})\geq\varphi(\{j\}),\quad \text{if } i\succsim j,\\ \varphi(\{i\})\geq\varphi(\{j\})+\varepsilon,\quad \text{if } i\succ j,\\ \varphi(\{i\})=\varphi(\{j\}),\quad \text{if } i\sim j,\\ \varphi(\{i,j\})\geq\varepsilon,\quad \text{if criteria } i \text{ and } j \text{ are synergic with } i,j\in G,\\ \varphi(\{i,j\})\leq-\varepsilon,\quad \text{if criteria } i \text{ and } j \text{ are redundant with } i,j\in G, \end{array}\right\}(E^{C})$$

- Constraints related to comparisons between alternatives,

$$\left.\begin{array}{l} C_\mu(a_k)\geq C_\mu(a_h),\quad \text{if } a_k\succsim a_h,\\ C_\mu(a_k)\geq C_\mu(a_h)+\varepsilon,\quad \text{if } a_k\succ a_h,\\ C_\mu(a_k)=C_\mu(a_h)\quad \text{if } a_k\sim a_h,\\ C_\mu(a_k)-C_\mu(a_h)\geq C_\mu(a_s)-C_\mu(a_t)+\varepsilon,\quad \text{if } (a_k,a_h)\succ^*(a_s,a_t),\\ C_\mu(a_k)-C_\mu(a_h)=C_\mu(a_s)-C_\mu(a_t),\quad \text{if } (a_k,a_h)\sim^*(a_s,a_t), \end{array}\right\}(E^{A})$$

where the strict inequalities used to translate the preferences have been transformed into weak inequalities in E^C and E^A by adding an auxiliary variable ε taking positive values.

We shall call *compatible model*, a capacity whose Möbius representation satisfies the set of constraints $E^{DM}=E^{MB}\cup E^C\cup E^A$ with a positive value of ε. Observe that E^C or E^A could be eventually empty if the DM does not provide any information on importance and interaction of criteria, or comparison of alternatives, respectively.

In order to check if there exists at least one compatible model, one has to solve the following linear programming problem:

$$\max \varepsilon = \varepsilon^* \quad s.t.$$
$$E^{DM}. \tag{3.3}$$

If E^{DM} is feasible and $\varepsilon^* > 0$, then there exists at least one model compatible with the preference information provided by the DM. If E^{DM} is infeasible or $\varepsilon^* \leq 0$, then one can check which is the minimum set of constraints determining the infeasibility using one of the techniques described in [125].

In this section, we shall describe how to obtain robust recommendations on the problem at hand by putting together the Choquet integral preference model and the SMAA methodology that is, by estimating the indices typical of SMAA, but considering as preference model the Choquet integral instead of an additive value function. We shall consider the following different cases:

case 1) the evaluations on the criteria are on a common scale and they are expressed in a precise way, that is $g_i(a_k) \in \mathbb{R}$ for all i and for all k,

case 2) the evaluations on criteria are on a common scale but they can be given in an imprecise way, that is $g_i(a_k) \in [\alpha_i^k, \beta_i^k]$ with $\alpha_i^k \leq \beta_i^k$, for some i and for some k,

case 3) the evaluations on the criteria are on different scales (for the sake of simplicity in this case we have supposed that evaluations of alternatives on the considered criteria are given in a precise way).

In **case 1)**, since the evaluations on the criteria under consideration are on a common scale and they are given in a precise way, the application of the Choquet integral depends only on a capacity compatible with the preferences expressed by the DM. Because the set of inequalities in E^{DM} defines a convex set of parameters, one can use the Hit-and-Run (HAR) method in order to sample some compatible models. The Hit-And-Run sampling has been firstly introduced in [158] and recently applied in multicriteria decision analysis in [168]. It starts from the choice of one point (the Möbius representation of one capacity in the problem at hand) inside the polytope E^{DM}. Since the starting point in the HAR sampling could be whichever point inside the polytope, we can begin from the point obtained by solving the linear optimization problem defined in (3.3). At each iteration, a random direction is sampled from the unit hypersphere that, passing through the starting point, generates a line. Finally, one point inside the segment whose extremes are the intersection of the line with the boundaries of the polytope is sampled.

Figure 3.1: Hit-and-Run example

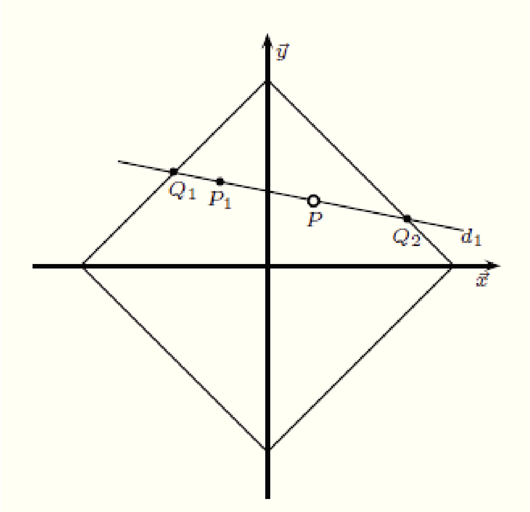


Figure 3.2: First iteration

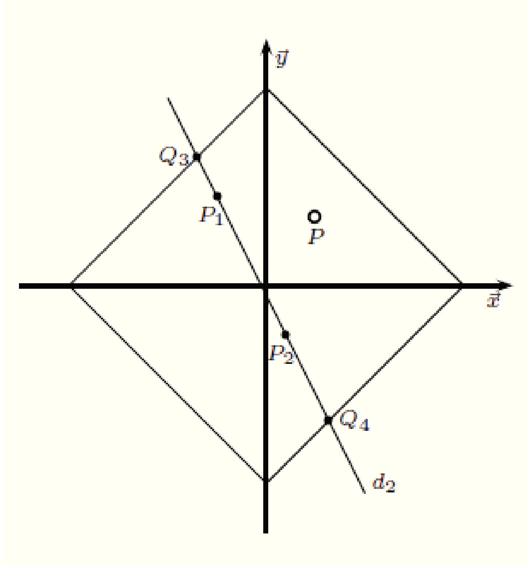


Figure 3.3: Second iteration

In order to illustrate the procedure, we shall provide the first two iterations of the Hit-and-Run algorithm in a didactic example. Let us suppose we have to sample some points (x, y) inside the region delimited by the constraints $y \leq x + 2$, $y \geq x - 2$, $y \leq -x + 2$ and $y \geq -x - 2$ (see Figure 3.1).

Chosen the starting point P and a vector belonging to the unit sphere of center $(0, 0)$ and radius equal to one that defines the direction d, we consider the line d_1 in Figure 3.2 having the direction of d and passing through the starting point P. d_1 "hits" the boundaries $y = x + 2$ and $y = -x + 2$ in the points Q_1 and Q_2, respectively. We then "run" along the segment Q_1Q_2, sampling in a uniform way the point P_1 at the first iteration. In the second iteration, the procedure continues considering P_1 as the starting point. Taking randomly a direction d, the line d_2 passing through point P_1 and having the same direction of d intersects the lines $y = x + 2$ and $y = x - 2$ in the points Q_3 and Q_4. Point P_2 is then chosen in a uniform way inside the segment connecting Q_3 and Q_4. The algorithm continues until the stopping rule (in our case the maximum number of iterations) is satisfied.

Let us observe that at each iteration of the HAR algorithm a compatible model is sampled and therefore stored. Consequently, by applying the Choquet integral preference model with each of the stored models, one can get one different ranking and, in the end, can estimate the indices typical of the SMAA methodology.

In **case 2)**, the application of the Choquet integral preference model does not depend on the sampled capacity only, but also on the evaluations of the alternatives at hand, because while con-

straints in E^C and in E^{MB} are not dependent on the alternatives' evaluations, constraints in E^A are dependent on these evaluations. Consequently, we have to distinguish between the case in which the DM does not provide any preference in terms of comparison between alternatives ($E^A = \emptyset$) from the case in which the DM expresses such type of preference ($E^A \neq \emptyset$).
If $E^A = \emptyset$, the set of constraints E^{DM} defines a convex set and therefore one can sample compatible models by applying the HAR method as described in the first case. The only difference with respect to **case 1)** is that, in order to apply the Choquet integral preference model, one has to sample an evaluation matrix M (whose element M_{ki} is taken in a random way inside the interval $[\alpha_i^k, \beta_i^k]$) for each stored capacity. After applying the Choquet integral preference model with the considered matrices and sampled capacities, one can compute the corresponding rankings and therefore estimating the SMAA indices.

Differently from the previous case, at each sampled evaluation matrix M corresponds a different set of constraints E^{DM}. Consequently, one can not apply the HAR method to sample the compatible capacities from E^A. Besides, sampled an evaluation matrix M, it is also possible that the corresponding set of constraints E^{DM} is infeasible. For this reason, after that an evaluation matrix has been sampled, one has to check if the set E^{DM} is feasible and, in this hypothesis, sampling a capacity compatible with the DM's preferences. Also in this case after storing the different rankings obtained by applying the Choquet integral with the sampled evaluations matrices and the corresponding sampled capacities, one can compute the SMAA indices.

A typical example of **case 3)** can be the evaluation of a sport car, where criteria such as maximum speed, acceleration, price, comfort can be considered and each of them has a different scale. In this case, one can not apply directly the Choquet integral to aggregate the preferences of the DM since, as remarked in the introduction, a requisite of the method is that all considered criteria are on a common scale.

In order to cope with this drawback, we propose to construct a common scale with a procedure composed of the following steps for each criterion g_i:

- sampling uniformly from the interval $[0, 1]$, l' different real numbers $x_1, \ldots, x_{l'}$ supposing that the different evaluations on g_i are l', with $l' \leq l$,
- ordering the l' numbers in an increasing way, $x_{i(1)} < \ldots < x_{i(l')}$,
- assigning $x_{i(h)}$ to the alternatives having the h-th evaluation, in an increasing order with respect to the DM's preferences on g_i.

Supposing to deal with the aforementioned car decision problem, and looking at the evaluations of the considered cars on criterion acceleration shown in Table 3.2, we proceed as follows:

- Because the evaluations of the 10 alternatives on criterion acceleration are all different, we sample 10 different real numbers from the interval $[0, 1]$. For example, $x_1 = 0.81$, $x_2 = 0.90$, $x_3 = 0.12$, $x_4 = 0.91$, $x_5 = 0.63$, $x_6 = 0.09$, $x_7 = 0.27$, $x_8 = 0.54$, $x_9 = 0.95$, $x_{10} = 0.96$.
- We order the 10 numbers in an increasing way: $x_{(1)} = 0.09 < x_{(2)} = 0.12 < x_{(3)} = 0.27 < x_{(4)} = 0.54 < x_{(5)} = 0.63 < x_{(6)} = 0.81 < x_{(7)} = 0.90 < x_{(8)} = 0.91 < x_{(9)} = 0.95 < x_{(10)} = 0.96$.
- Since, in this example, acceleration has a decreasing direction of preference (the lower the evaluation on the criterion, the better the alternative is) we assign value $x_{(1)} = 0.09$ to SEAT Ibiza ST 1.2, value $x_{(2)} = 0.12$ to SKODA Fabia 1.2 and so on (see the third column of Table 3.2).

Table 3.2: Car evaluation with respect to the criterion acceleration (expressed in seconds necessary to reach 100 Km/h starting from 0 Km/h) and the corresponding scale

Cars	Acceleration 0/100 km/h	Scale value
PEUGEOT 208 1.6 8V	10.9	0.96
Citroen C3	13.5	0.54
FIAT 500 0.9	11	0.95
SKODA Fabia 1.2	14.2	0.12
LANCIA Ypsilon 5p	11.4	0.90
RENAULT Clio 1.5 dCi 90	11.3	0.91
SEAT Ibiza ST 1.2	14.6	0.09
ALFA ROMEO MiTo 1.3	12.9	0.63
TOYOTA Yaris 1.5	11.8	0.81
VOLKSWAGEN Polo 1.2	13.9	0.27

The values $x_{i(r)}, i = 1, \ldots, n$ and $r = 1, \ldots, l'$, become the evaluations of the considered alternatives on the different criteria. In this way, evaluations on all criteria are expressed on the same common scale and therefore, having a capacity compatible with the DM's preferences, one can compute the Choquet integral of all alternatives.

At this point, since the sampling of a compatible model will depend on the chosen common scale only, if $E^A \neq \emptyset$ (the DM provides some preference on the considered alternatives), one can proceed as already described in **case 2)**, but replacing the sampling of an evaluation matrix with the construction of a common scale. The only difference with **case 2)** is that the DM could be interested in discovering which is the most discriminant common scale. With this aim, one can proceed as follows:

- Sampling a certain number of possible common scales $S_1, \ldots, S_{iter}$, considering the corresponding feasible sets of constraints $E_1^{DM}, \ldots, E_{iter}^{DM}$ and denoted by $\varepsilon_1, \ldots, \varepsilon_{iter}$, the solutions of the linear programming problems

$$\left.\begin{array}{ll}\max \varepsilon & s.t. \\ E_1^{DM} & \end{array}\right\}, \ldots\ldots\ldots\ldots, \quad \left.\begin{array}{ll}\max \varepsilon & s.t. \\ E_{iter}^{DM} & \end{array}\right\} \tag{3.4}$$

the most discriminant scale is the scale S_k such that $\varepsilon_k = \max\{\varepsilon_1, \ldots, \varepsilon_{iter}\}$.

After obtaining the most discriminant common scale, the decision aiding process can continue in one of the following ways:

- applying the Choquet integral preference model after asking the capacities directly to the DM,
- eliciting one (arbitrary) capacity compatible with the DM's preference information [119],
- considering the whole set of capacities compatible with the DM's preference information using NAROR [8],
- applying the simulation techniques proposed in **case 1)** since the common scale's values become the evaluations of the alternatives on the considered criteria.

3.1.5 Some examples

The whole methodology presented in the previous section will be illustrated by two didactic examples. In the following, we shall consider uniform probability distributions f_W and f_χ, respectively, on W and χ.

Considering imprecision in the evaluations on considered criteria

Let us consider a set of 18 alternatives evaluated on the basis of 4 criteria, $G = \{g_1, g_2, g_3, g_4\}$, as shown in Table 3.3. We suppose that the evaluations of considered alternatives on each criterion are integer numbers within an interval (for example, the evaluation of a_1 on criterion g_1 can be 14, 15 or 16), but this is not a specific requirement for our model i.e., in general, we can sample values from the whole interval. We can consider this as a specific probability distribution $f_\chi(\xi)$ concentrating uniformly the mass only on the integers in the interval of evaluations on considered criteria.

We shall take into account the following preference information in terms of importance and interaction of criteria and comparisons between alternatives:

Table 3.3: Imprecise evaluations of alternatives on considered criteria

	Alternatives								
Criteria	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
g_1	[14,16]	[6,8]	[17,19]	[8,10]	[11,13]	[7,9]	[13,15]	[7,9]	3
g_2	[11,13]	[7,9]	[7,9]	[15,17]	5	3	[18,20]	[12,14]	[16,18]
g_3	[9,11]	[13,15]	4	[3,5]	[13,15]	[6,9]	5	[14,15]	2
g_4	[6,9]	[15,17]	[11,13]	[15,17]	[13,15]	[18,20]	[9,11]	6	[13,15]
Criteria	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}	a_{18}
g_1	4	[15,17]	[7,9]	[16,18]	[7,9]	[18,20]	[11,13]	[13,15]	[8,10]
g_2	[18,20]	7	[10,12]	[11,13]	[6,8]	[6,9]	4	[10,12]	[12,14]
g_3	[7,9]	[13,15]	5	[5,7]	[6,9]	[3,5]	[14,16]	[11,13]	[11,13]
g_4	[8,10]	[9,11]	[18,20]	8	[18,20]	[11,13]	[12,15]	[8,10]	[5,7]

- $\varphi(\{g_1\} > \varphi(\{g_2\}), \varphi(\{g_3\} > \varphi(\{g_4\}),$
- $\varphi(\{g_1, g_2\}) > 0$, $\varphi(\{g_2, g_3\}) > 0$, $\varphi(\{g_2, g_4\}) < 0$,
- $a_{16} \succ a_2$, $a_3 \succ a_{14}$ and $a_{13} \succ a_8$.

According to [167], we perform the Hit-and-Run procedure for $10,000$ iterations.
For each iteration, we sample an evaluation matrix and we check if it is compatible with the preference information provided by the DM. In this case, we compute the Choquet integral for each alternative obtaining a complete ranking.
At the end of all iterations, we compute the rank acceptability index b_k^r for each $k, r = 1, \ldots, l$ and the Möbius representation of the central capacity for each alternative a_k that can get the first rank at least once, as shown, respectively, in Tables 3.4 and 3.5. In particular, in Table 3.4 we observe that alternatives a_1, a_3, a_7, a_{11}, a_{13}, a_{15}, a_{16} and a_{17} can be ranked first. a_{17} has reached the first position more than all other alternatives ($b_{17}^1 = 25.39$) and a_9 is instead the alternative that is almost always in the last position in the obtained rankings ($b_9^{18} = 99.52$).

Looking at the second best alternative, one can be in doubt among a_{11}, a_7 and a_1. In fact, on one side a_7 has a first rank acceptability index greater than the other two alternatives ($b_7^1 = 24.68\%$). On the other side, looking at the pairwise winning indices shown in Table 3.6, one can observe that a_{11} and a_1 are weakly preferred to all other alternatives with a frequency of at least 47.04% and 44.09%, respectively (vs the 40.20% of a_7) and both of them are preferred to a_7 more frequently than the viceversa. At the same time, a_9 can be considered surely the worst alternative because all alternatives are weakly preferred to it with a frequency at least equal to the 99.54%.

Computing the Möbius representation of the barycenter of compatible capacities shown in Table 3.7 and applying the Choquet integral to the average evaluation matrix we get the following ranking

Table 3.4: Rank acceptability indices taking into account imprecise evaluations of alternatives on considered criteria, preference information in terms of importance and interaction of criteria and comparisons between alternatives

Alt	b_k^1	b_k^2	b_k^3	b_k^4	b_k^5	b_k^6	b_k^7	b_k^8	b_k^9	b_k^{10}	b_k^{11}	b_k^{12}	b_k^{13}	b_k^{14}	b_k^{15}	b_k^{16}	b_k^{17}	b_k^{18}
a_1	19.69	23.35	22.59	15.98	8.81	5.08	2.51	1.08	0.63	0.21	0.06	0	0.01	0	0	0	0	0
a_2	0	0.02	0.03	0.06	0.34	1.52	4.23	7.25	11.2	15.38	17.62	18.27	12.54	7.19	3.76	0.55	0.04	0
a_3	0.59	1.2	2.63	3.16	6.11	9.72	13.18	13.16	13.09	12.37	11.4	9.73	2.84	0.73	0.06	0.03	0	0
a_4	0	0.01	0.01	0.07	0.11	0.43	1.27	2.38	3.16	4.9	6.28	8.5	14.2	21.1	24.08	11.69	1.81	0
a_5	0.6	1.64	3.62	6.82	10.98	12.48	12.82	12.96	12.69	9.64	7.66	4.64	2.31	0.9	0.23	0.01	0	0
a_6	0	0	0	0	0.01	0	0.03	0.04	0.03	0.18	0.32	1.05	2.2	5.32	10.87	40.02	39.47	0.46
a_7	24.68	15.79	15.99	18	10.99	6.34	3.14	2.35	1.49	0.72	0.35	0.07	0.06	0.03	0	0	0	0
a_8	0	0	0.08	0.43	1.77	7.96	11.39	13.81	12.24	12.92	12.12	9.81	7.74	5.29	3.45	0.97	0.02	0
a_9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.01	0.47	99.52
a_{10}	0	0	0	0	0	0	0.01	0	0.02	0.02	0.12	0.2	0.66	2.32	6.87	32.5	57.26	0.02
a_{11}	23.82	21.83	17.88	15.05	9.89	5.76	3.6	1.69	0.37	0.05	0.06	0	0	0	0	0	0	0
a_{12}	0	0	0.01	0	0.01	0.11	0.25	0.64	1.85	3.35	5.44	9.08	16.84	26.49	27.61	7.86	0.46	0
a_{13}	2.27	6.37	10.45	15.96	24.34	17.15	11.08	6.36	3.79	1.47	0.57	0.15	0.03	0.01	0	0	0	0
a_{14}	0	0	0	0	0	0.05	0.34	1.31	3.06	5.57	8.87	13.99	24.1	20.09	17.52	4.76	0.34	0
a_{15}	2.49	4	4.99	7.5	9.91	13.23	12.67	10.32	8.65	6.92	6.43	5.88	3.27	2.18	1.05	0.4	0.11	0
a_{16}	0.47	1.16	1.65	2.9	5.14	9.15	12.77	16.41	16.85	14.64	10.26	5.19	2.36	1	0.05	0	0	0
a_{17}	25.39	24.59	19.83	12.91	7.95	4.45	2.96	1.19	0.58	0.13	0.02	0	0	0	0	0	0	0
a_{18}	0	0.04	0.24	1.16	3.64	6.57	7.75	9.05	10.3	11.53	12.42	13.44	10.84	7.35	4.45	1.2	0.02	0

Table 3.5: Möbius representation of central capacities for alternatives taking into account imprecise evaluations of alternatives on considered criteria, preferences on importance and interaction of criteria and comparisons between alternatives

Alt/Möbius	$m(\{1\})$	$m(\{2\})$	$m(\{3\})$	$m(\{4\})$	$m(\{1,2\})$	$m(\{1,3\})$	$m(\{1,4\})$	$m(\{2,3\})$	$m(\{2,4\})$	$m(\{3,4\})$
a_1	0.31	0.19	0.18	0.19	0.10	0.03	0.00	0.10	-0.10	-0.00
a_3	0.41	0.16	0.25	0.21	0.08	-0.11	0.02	0.08	-0.08	-0.03
a_5	0.29	0.16	0.20	0.22	0.07	0.03	0.04	0.07	-0.11	0.03
a_7	0.32	0.21	0.21	0.19	0.11	-0.04	0.01	0.09	-0.09	-0.01
a_{11}	0.30	0.17	0.16	0.19	0.08	0.10	0.01	0.08	-0.09	-0.00
a_{13}	0.34	0.19	0.21	0.19	0.11	-0.04	0.00	0.09	-0.10	-0.01
a_{15}	0.39	0.17	0.25	0.21	0.09	-0.10	0.02	0.09	-0.09	-0.03
a_{16}	0.32	0.16	0.25	0.21	0.05	-0.04	0.05	0.06	-0.11	0.05
a_{17}	0.29	0.19	0.17	0.19	0.10	0.07	0.00	0.10	-0.10	-0.00

of the considered alternatives:

$$a_{17} \succ a_{11} \succ a_1 \succ a_7 \succ a_{13} \succ a_{15} \succ a_5 \succ a_3 \succ a_{16} \succ a_8 \succ a_{18} \succ a_2 \succ a_{14} \succ a_{12} \succ a_4 \succ a_6 \succ a_{10} \succ a_9.$$

An example with the criteria expressed on different scales

In this section, we deal with a decision making problem in which the evaluation of alternatives on considered criteria are expressed on heterogeneous scales.

From the city-car segment market, we select ten cars evaluated on the basis of the following criteria: price (in Euro), acceleration (0/100 km/h in seconds), maximum speed (in km/h) and consumption (in l/100km) (see Table 3.8). In this example, we shall suppose that price, acceleration and consumption have a decreasing direction of preference (the lower the evaluation of an alternative on the

Table 3.6: Pairwise winning indices taking into account imprecise evaluations of alternatives on considered criteria, preferences on importance and interaction of criteria and comparisons between alternatives

Alt/Alt	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}	a_{18}
a_1	0.00	99.09	92.23	99.70	91.39	99.98	55.80	98.53	100.00	100.00	47.59	99.88	80.36	99.85	85.65	94.32	44.09	99.16
a_2	0.91	0.00	29.52	78.07	14.31	99.20	2.96	37.69	99.99	99.33	0.25	83.54	6.58	77.66	23.08	0.00	0.39	45.69
a_3	7.77	70.48	0.00	93.07	43.69	99.57	8.24	61.99	100.00	99.87	8.77	95.18	18.31	100.00	35.82	53.00	8.02	66.50
a_4	0.30	21.93	6.93	0.00	9.17	86.72	0.23	16.44	100.00	95.37	0.45	51.96	0.97	42.25	6.56	11.77	0.23	21.00
a_5	8.61	85.69	56.31	90.83	0.00	99.99	15.78	65.84	100.00	99.85	2.72	94.92	27.39	94.32	46.07	62.74	5.17	73.95
a_6	0.02	0.80	0.43	13.28	0.01	0.00	0.14	2.68	99.59	62.13	0.00	9.16	0.13	4.64	0.78	0.04	0.00	2.85
a_7	44.20	97.04	91.76	99.77	84.22	99.86	0.00	96.75	100.00	100.00	43.89	99.73	76.56	99.37	84.76	90.05	40.20	96.87
a_8	1.47	62.31	38.01	83.56	34.16	97.32	3.25	0.00	100.00	99.93	2.64	86.11	0.00	79.90	29.48	42.28	1.66	58.17
a_9	0.00	0.01	0.00	0.00	0.00	0.41	0.00	0.00	0.00	0.20	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_{10}	0.00	0.67	0.13	4.63	0.15	37.87	0.00	0.07	99.80	0.00	0.00	4.19	0.00	3.89	0.28	0.19	0.00	0.19
a_{11}	52.41	99.75	91.23	99.55	97.28	100.00	56.11	97.36	100.00	100.00	0.00	99.94	77.93	99.96	84.83	97.71	47.04	98.97
a_{12}	0.12	16.46	4.82	48.04	5.08	90.84	0.27	13.89	100.00	95.81	0.06	0.00	0.60	39.11	5.38	7.14	0.04	17.50
a_{13}	19.64	93.42	81.69	99.03	72.61	99.87	23.44	100.00	100.00	100.00	22.07	99.40	0.00	99.07	69.73	82.77	18.51	91.79
a_{14}	0.15	22.34	0.00	57.75	5.68	95.36	0.63	20.10	100.00	96.11	0.04	60.89	0.93	0.00	7.73	9.83	0.05	24.04
a_{15}	14.35	76.92	63.57	93.44	53.93	99.22	15.24	70.52	100.00	99.72	15.17	94.62	30.27	92.27	0.00	63.15	14.14	72.99
a_{16}	5.68	100.00	47.00	88.23	37.26	99.96	9.95	57.72	100.00	99.81	2.29	92.86	17.23	90.17	36.85	0.00	4.06	65.09
a_{17}	55.82	99.61	91.98	99.77	94.83	100.00	59.80	98.34	100.00	100.00	52.96	99.96	81.49	99.95	85.86	95.94	0.00	99.46
a_{18}	0.84	54.31	33.50	79.00	26.05	97.15	3.13	41.83	100.00	99.81	1.03	82.50	8.21	75.96	27.01	34.91	0.54	0.00

Table 3.7: Möbius representation of the barycenter of the compatible capacities taking into account interval evaluations of alternatives on considered criteria, preference information on importance and interaction of criteria and comparisons between alternatives

$m(\{1\})$	$m(\{2\})$	$m(\{3\})$	$m(\{4\})$	$m(\{1,2\})$	$m(\{1,3\})$	$m(\{1,4\})$	$m(\{2,3\})$	$m(\{2,4\})$	$m(\{3,4\})$
0.31	0.18	0.19	0.19	0.097	0.034	0.008	0.091	-0.08	-0.014

criterion, the better the alternative is on the considered criterion), while criterion maximum speed has an increasing direction of preference (the higher the evaluation of an alternative on a criterion, the better the alternative is on the considered criterion). Let us notice that, in some cases, criteria are non monotonic with respect to the preferences of the DM. This means that one can not state that the criterion has a decreasing or an increasing direction of preference.

Let suppose that the DM supplies the following preference information in terms of importance and interaction of criteria as well as in terms of comparisons between alternatives:

- $\varphi(\{g_1\}) > \varphi(\{g_2\})$, $\varphi(\{g_4\}) > \varphi(\{g_3\})$,
- $\varphi(\{g_3, g_4\}) > 0$, $\varphi(\{g_2, g_3\}) < 0$.
- $a_5 \succ a_1$, $a_7 \succ a_6$, $a_2 \succ a_3$,

As explained in Section 3.1.4, at each iteration we sample a common scale, and, if the set of constraints E^{DM} is feasible, then we sample a capacity compatible with these constraints. Let us notice that since the DM has provided some preference in terms of comparison between alternatives, the set of constraints E^{DM} will be dependent on the sampled scale.

At the end of all the iterations, we shall get the rank acceptability indices, the Möbius representations

Table 3.8: Evaluation matrix

Cars	Price Euro	Acceleration 0/100 km/h	Max speed km/h	Consumption l/100km
PEUGEOT 208 1.6 8V e-HDi 92 CV Stop&Start 3p. Allure	17,800	10.9	185	3.8
Citroen C3 1.4 HDi 70 Seduction	15,750	13.5	163	3.8
FIAT 500 0.9 TwinAir Turbo Street	15,050	11	173	4
SKODA Fabia 1.2 TDI CR 75 CV 5p. GreenLine	15,260	14.2	172	3.4
LANCIA Ypsilon 5p 1.3 MJT 95 CV 5p. S&S Gold	16,300	11.4	183	3.8
RENAULT Clio 1.5 dCi 90 CV 3p. Dynamique	16,050	11.3	176	4
SEAT Ibiza ST 1.2 TDI CR Ecomotive	15,700	14.6	173	3.4
ALFA ROMEO MiTo 1.3 JTDm 85 CV S&S Progression	17,500	12.9	174	3.5
TOYOTA Yaris 1.5 Hybrid 5p. Lounge	17,800	11.8	165	3.2
VOLKSWAGEN Polo 1.2 TDI 5p. BlueMotion 89g	17,060	13.9	173	3.4

of the central capacities for each alternative and the preference matrix shown respectively in Tables 3.10, 3.11 and 3.12 in the Appendix.

In Table 3.10, we observe that car a_4 is the most preferred by the DM ($b_4^1 = 55.51\%$) followed by a_7, while a_6 is most frequently the least preferred car ($b_6^{10} = 53.04\%$) and a_1, a_2, a_3 and a_6 can never arrive first. Table 3.11 gives the Möbius representations of the central capacities ranking considered alternatives in the first position at least once, while from Table 3.12, giving the frequency of the weak preference between pairs of alternatives, we observe that a_4 is weakly preferred to all other alternatives with a frequency at least equal to 67.71%.

Since there are different common compatible scales, we propose the most discriminant common scale, presented in Table 3.9, to the DM.

Table 3.9: Evaluations of alternatives on considered criteria expressed on the most discriminating common scale

Alt	Price	Acceleration	Max speed	Consumption
a_1	0.1834	0.7290	0.8208	0.5723
a_2	0.5870	0.4023	0.2107	0.5723
a_3	0.8663	0.6981	0.4427	0.0496
a_4	0.8567	0.1268	0.4234	0.7090
a_5	0.5613	0.5854	0.6979	0.5723
a_6	0.5721	0.6626	0.5906	0.0496
a_7	0.7443	0.0569	0.4427	0.7090
a_8	0.3115	0.4438	0.5717	0.6015
a_9	0.1834	0.5816	0.3944	0.8207
a_{10}	0.4113	0.3501	0.4427	0.7090

After the DM accepts the common scale, we apply SMAA sampling capacities compatible with the preference information provided by the DM, computing the rank acceptability indices, the Möbius representations of the central capacities and the preference matrix displayed, respectively, in Tables 3.13, 3.14 and 3.16 in the Appendix. Applying the Choquet integral with respect to the barycenter of the compatible capacities whose Möbius representation are shown in Table 3.15, and considering the most discriminating common scale we get the following ranking of the considered alternatives:

$$a_5 \succ a_4 \succ a_7 \succ a_1 \succ a_{10} \succ a_8 \succ a_2 \succ a_9 \succ a_3 \succ a_6.$$

3.1.6 Conclusions

In this paper, we have combined the Stochastic Multiobjective Acceptability Analysis (SMAA) to the Choquet integral preference model extending a work already published by the authors [3]. We have proposed to explore the space of the parameters compatible with some preference information provided by the DM using SMAA. In particular, we have considered the DM's preference information not only in terms of relative importance of criteria and interaction between them, but differently from [3], also in terms of pairwise comparison between alternatives and comparisons of intensity of preferences between pairs of alternatives. Moreover, again differently from [3], we have considered also imprecise evaluations of alternatives on the considered criteria.

Finally, we have proposed a methodology to construct the common scale required by the Choquet integral; this is very useful in case the criteria for the decision problem at hand are defined on different scales. Such aspect of the methodology we are proposing constitutes another original contribution with respect to [3]. We have provided some didactic examples in which the proposed methodology has been applied. We envisage the following future developments:

- application of SMAA methodology to some extensions of the classical Choquet integral, e.g. the bipolar Choquet integral [71, 72], the level dependent Choquet integral [80], the robust Choquet integral [90];
- implementation of the SMAA methodology to the Choquet integral in presence of hierarchy of criteria [5] within the so called multiple criteria hierarchy process [37].

Appendix

Table 3.10: Rank acceptability indices sampling simultaneously compatible capacities and scales

Alt	b_k^1	b_k^2	b_k^3	b_k^4	b_k^5	b_k^6	b_k^7	b_k^8	b_k^9	b_k^{10}
a_1	0	0.08	0.1	1.61	5.03	18.28	32.05	26.28	4.07	12.5
a_2	0	0.02	1.9	4.12	4.99	14.82	45.23	26.46	2.46	0
a_3	0	0	0	0.57	0.71	1.02	2.98	15.04	52.03	27.65
a_4	55.51	32.37	7.54	3.45	0.99	0.1	0.04	0	0	0
a_5	4.45	2.78	18.38	12.17	23.54	36.88	1.14	0.47	0.19	0
a_6	0	0	0	0.05	0.41	0.89	1.91	10.46	33.24	53.04
a_7	26.15	53.89	11.96	5.46	2.29	0.19	0.06	0	0	0
a_8	4.79	4	20.66	35.32	23.96	6.21	2.88	1.27	0.81	0.1
a_9	5.9	1.72	8.38	9.51	20.36	15.11	9.08	17.92	5.83	6.19
a_{10}	3.2	5.14	31.08	27.74	17.72	6.5	4.63	2.1	1.37	0.52

Table 3.11: Möbius representations of central capacities sampling simultaneously compatible capacities and scales

Alt/Möbius	$m(\{1\})$	$m(\{2\})$	$m(\{3\})$	$m(\{4\})$	$m(\{1,2\})$	$m(\{1,3\})$	$m(\{1,4\})$	$m(\{2,3\})$	$m(\{2,4\})$	$m(\{3,4\})$
a_4	0.06	0.09	0.09	0.17	0.00	0.01	0.48	-0.05	0.08	0.06
a_5	0.09	0.17	0.17	0.16	0.00	0.01	0.38	-0.09	0.02	0.09
a_7	0.06	0.10	0.10	0.18	0.00	0.01	0.48	-0.05	0.05	0.07
a_8	0.06	0.10	0.10	0.17	0.00	0.02	0.41	-0.05	0.11	0.09
a_9	0.05	0.04	0.04	0.28	0.00	0.03	0.34	-0.02	0.17	0.05
a_{10}	0.04	0.07	0.07	0.17	0.00	0.01	0.46	-0.04	0.14	0.08

3.2 Combining Analytical Hierarchy Process and Choquet integral within Non Addititive Robust Ordinal Regression

3.2.1 Introduction

In Multiple Criteria Decision Aiding (MCDA) problems (see [95] for an accessible guide to MCDA and [51] for a comprehensive collection of state of the art surveys), a set of alternatives $A = \{a, b, c, \ldots\}$ is evaluated on a set of evaluation criteria $G = \{g_1, \ldots, g_n\}$ (sometimes, for the sake of simplicity and slightly abusing of the notation, we refer to the criteria with their indices, i.e. we shall write $i \in G$, instead of $g_i \in G$). Typical MCDA problems are choice, sorting and ranking. Choice problems consist of choosing a subset (possibly composed of one element only) $A^* \subseteq A$ of alternatives considered the best; sorting problems consist of assigning each alternative to one or more predefined and preferentially ordered contiguous classes, while ranking problems consist of partially or completely ordering all alternatives from the best to the worst.

Looking at the evaluations of the alternatives on the criteria, without taking into account further preference information and any preference model, it could be only observed if the dominance rela-

Table 3.12: Pairwise winning indices considering a simulation sampling of random capacities and common scales

Alt/Alt	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
a_1	0	44.25	82.51	0.27	0	85.29	0.32	5.45	34.33	9.63
a_2	55.75	0	100	0.07	4.04	97.45	0.09	9.72	35.01	11.35
a_3	17.49	0	0	0	1.37	65.79	0	2.81	11.38	3.26
a_4	99.73	99.93	100	0	93.99	100	67.71	91.54	93.13	91.47
a_5	100	95.96	98.63	6.01	0	99.45	7.36	34.36	58.28	33.69
a_6	14.71	2.55	34.21	0	0.55	0	0	1.94	9.26	2.58
a_7	99.68	99.91	100	32.29	92.64	100	0	89.46	91.59	89.77
a_8	94.55	90.28	97.19	8.46	65.64	98.06	10.54	0	75.23	48.33
a_9	65.67	64.99	88.62	6.87	41.72	90.74	8.41	24.77	0	21.94
a_{10}	90.37	88.65	96.74	8.53	66.31	97.42	10.23	51.67	78.06	0

Table 3.13: Rank acceptability indices taking into account evaluations of alternatives on considered criteria expressed on the most discriminating common scale shown in Table 3.9

Alt	b_k^1	b_k^2	b_k^3	b_k^4	b_k^5	b_k^6	b_k^7	b_k^8	b_k^9	b_k^{10}
a_1	0	17.76	8.48	22	10.92	15.65	12.82	9.17	2.42	0.78
a_2	0	2.23	5.39	19.02	12.61	7.36	12.66	39.6	1.13	0
a_3	0	0	0.33	1.53	2.79	3.18	2.93	7.79	69.46	11.99
a_4	32.28	41.1	14.26	6.91	3.2	2	0.25	0	0	0
a_5	65.88	12.06	20.58	1.24	0.24	0	0	0	0	0
a_6	0	0	0	0	0	0.01	0.73	2.05	14.97	82.24
a_7	0.81	21.11	39	15.1	8.06	5.53	5.91	2.38	2.1	0
a_8	0	0.35	4.3	5.79	21.08	28.94	29.92	8.89	0.73	0
a_9	1.03	2.4	5.23	7.01	15.7	9.83	18.64	26.38	8.79	4.99
a_{10}	0	2.99	2.43	21.4	25.4	27.5	16.14	3.74	0.4	0

tion is fulfilled by some pairs of alternatives [1]. In general, the dominance relation provides really poor information and leaves many alternatives incomparable. For this reason, to get more precise recommendations on the problem at hand, there is the necessity to aggregate the evaluations of the alternatives on the considered criteria through some appropriate preference model representing the preferences of the Decision Maker (DM). In the literature the most well-known aggregation methods are the Multi-Attribute Value Theory (MAVT) [109] and the outranking methods (for ELECTRE methods see [55, 54, 143] and for PROMETHEE methods see [26, 27]). MAVT assigns to each al-

[1] An alternative a dominates an alternative b if the evaluations of a are at least as good as the evaluations of b on all criteria and better for at least one criterion.

Table 3.14: Möbius representation of the central capacities, taking into account evaluations of alternatives on considered criteria expressed on the most discriminating common scale, shown in Table 3.9

Alt/Möbius	$m(\{1\})$	$m(\{2\})$	$m(\{3\})$	$m(\{4\})$	$m(\{1,2\})$	$m(\{1,3\})$	$m(\{1,4\})$	$m(\{2,3\})$	$m(\{2,4\})$	$m(\{3,4\})$
a_4	0.21	0.18	0.15	0.31	-0.01	0.02	0.17	-0.06	-0.03	0.08
a_5	0.15	0.16	0.16	0.18	0.04	0.03	0.14	-0.06	0.07	0.12
a_7	0.03	0.26	0.16	0.30	-0.01	0.07	0.38	-0.11	-0.13	0.04
a_9	0.24	0.15	0.16	0.46	-0.03	0.03	-0.11	-0.07	0.11	0.07

Table 3.15: Möbius representation of the barycenter of capacities taking into account evaluations of alternatives on the considered criteria expressed on the most discriminant common scale shown in Table 3.9

$m(\{1\})$	$m(\{2\})$	$m(\{3\})$	$m(\{4\})$	$m(\{1,2\})$	$m(\{1,3\})$	$m(\{1,4\})$	$m(\{2,3\})$	$m(\{2,4\})$	$m(\{3,4\})$
0.17	0.17	0.16	0.23	0.02	0.03	0.15	-0.06	0.04	0.10

Table 3.16: Pairwise winning indices taking into account evaluations of alternatives on the most discriminant common scale shown in Table 3.9

Alt/Alt	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
a_1	0	64.01	92.08	20.67	0	97.97	31.4	71.33	77.54	54.86
a_2	35.99	0	100	4.12	0	98.87	11.73	44.58	50.08	35.12
a_3	7.92	0	0	0.82	0	87.97	3.86	8.43	16.46	6.53
a_4	79.33	95.88	99.18	0	33.34	100	97.25	92.89	93.26	94.22
a_5	100	100	100	66.66	0	100	78.61	100	97.19	99.64
a_6	2.03	1.13	12.03	0	0	0	0	0.74	5.36	0.01
a_7	68.6	88.27	96.14	2.75	21.39	100	0	83.14	82.55	83.94
a_8	28.67	55.42	91.57	7.11	0	99.26	16.86	0	64.53	33.65
a_9	22.46	49.92	83.54	6.74	2.81	94.64	17.45	35.47	0	29.4
a_{10}	45.14	64.88	93.47	5.78	0.36	99.99	16.06	66.35	70.6	0

ternative a a real number $U(a)$ being representative of the degree of desirability of a with respect to the problem at hand, while outranking methods are based on an outranking relation being a binary relation S on the set of alternatives A, such that aSb means that a is at least as good as b.

Both family of methods are based on the mutual preference independence between criteria [109, 174] but, in many real world decision making problems, the evaluation criteria are not independent but interacting. For instance, suppose the DM likes sport cars and she wants to buy a car taking into account the criteria price, maximum speed and acceleration. In this case, maximum speed and acceleration can be considered negatively interacting criteria while maximum speed and price can be considered positively interacting criteria. In fact, on one hand, even if maximum speed and acceleration are very important for a DM liking sport cars, in general cars with a high maximum speed have also a good acceleration and, therefore, the comprehensive importance of the two criteria considered together should be smaller than the sum of the importance of the two criteria considered alone. On the other hand, a car with a high maximum speed has often also a high price and, therefore, a car with a high maximum speed and a moderate price is very well appreciated. Consequently, the comprehensive importance of these two criteria considered together should be greater than the sum of the importance of the two criteria considered alone.

In such cases, the mutual preference independence can be violated because, for example, due to the positive interaction between maximum speed and price, at a given level of price, one can prefer one combination of maximum speed and acceleration, while at another level of price one can prefer another combination of maximum speed and acceleration. Observe however that the violation of preference independence does not imply that the considered family of criteria is no more consistent. Indeed, consistency [143] refers to the requirements of *monotonicity*, that is, when improving the evaluations on considered criteria the overall evaluation of an alternative cannot be deteriorated, *exhaustivity*, that is, all the relevant criteria are considered, and *non-redundancy*, that is, no criterion can be removed without losing the representation of a relevant point of view. Monotonicity, exhaustivity and non-redundancy can continue to be satisfied also when preference independence does not hold. For instance, in the didactic example of Section 2, we show how reasonable can be the overall evaluations of students obtained aggregating scores in different subjects by the Choquet integral [31] rather than by the weighted sum. If the problem is correctly formulated, aggregation through Choquet integral satisfies monotonicity, exhaustivity and non-redundancy even if, as explained in the example, it does not satisfy preference independence.

Interaction between criteria and violation of the preference independence are well known in MCDA (see e.g. [13, 50, 57, 108]). In the following, we briefly survey several methods handling with the interaction between criteria. Considering the utility functions as aggregation methods, the multilinear utility function [109] and the UTA^{GMS}-INT [88] are reported in the literature. The first one aggregates performances on considered criteria through a weighted sum of products of marginal utilities corresponding to single criteria, over all subsets of criteria, while UTA^{GMS}-INT is based on enriched additive value functions that add some further terms representing interaction between criteria to the usual sum of marginal utility functions. In Artificial Intelligence (AI), interaction between criteria has been recently considered through GAI-networks [67] as well as through UCP-networks [17], that are based on the idea of Generalized Additive Independence (GAI) decomposition [56]. Positive and negative interaction between criteria has been taken into account also in outranking methods such as ELECTRE [52] and PROMETHEE [34]. Another method that takes into consideration interaction between criteria is the Analytical Network Process (ANP) [147]. In this case, interaction between criteria is one of the possible results of interdependencies and network between goals, criteria and alternatives. Observe that very specific interactions between criteria can be considered within ANP. For instance, ANP can model interactions that depend on the considered alternatives. This is the case of a positive interaction between criteria "price" and "maximum speed" for evaluating an economic car, which is not the case for a sport car. Considering interaction between criteria that can change from an alternative to another is not possible with the Choquet integral for which interaction between criteria holds in the same way for all the alternatives. However, the price to pay for such so fine modeling is an increased amount of preference information that can be difficult to supply for the DM.

Even if all cited methods are able to deal with the interaction between criteria, the most well-known methodologies in the literature are the non-additive integrals, such as the Choquet integral (see [31] for the original Choquet integral and [68] for the application of the Choquet integral in MCDA), the Sugeno integral [162] and the generalizations of the Choquet integral, that are the bipolar Choquet integral [71] or the level dependent Choquet integral[80]. The basic idea of these approaches is that the interaction between criteria can be modeled through a capacity, called also fuzzy measure, assigning a weight not only to each criterion but also to each subset of criteria.

In this paper, we shall consider the Choquet integral because, currently, it is the most adopted methodology to deal with interactions between criteria for its manageability (for example, we shall see that we can use linear programming to determine capacities compatible with DM's preferences) and for the meaningfulness of its preference parameters, namely the capacity that becomes understandable and intelligible even for the non expert DM using some specific techniques such as the Möbius representation, the Shapley index and the interaction indices.

Even if it is theoretically appealing, the application of the Choquet integral, as well as the application of all non-additive methods mentioned above, involves some problems related to:

1) the determination of the capacity representing the interaction between criteria,

2) the construction of a common scale permitting comparisons between evaluations on different criteria.

To handle point 1), we propose to use the Non Additive Robust Ordinal Regression (NAROR) [8] that considers the whole set of capacities compatible with the preference information provided by the DM while, to handle point 2) we propose to use the Analytic Hierarchy Process (AHP, [145, 146]). Let us spend some words to give the intuition behind our proposal. We shall give more details on how to deal with the two mentioned problems and on the reasons of combining them together in the following sections of the paper.

In any MCDA problem, a decision model has to be built to produce a recommendation and its preference parameters (weights, thresholds, value functions and so on) have to be determined. This is usually done in cooperation with the DM, who can give directly the preference parameters or, instead, can supply some preference information, for example in terms of preference pairwise comparison of some alternatives, from which preference parameters can be induced. In the case of the Choquet integral model, the preference parameters to be fixed are the weights that the capacity assigns to each one of the 2^n subsets of a family of n criteria (for example $2^{10} = 1024$ weights for a family of 10 criteria). Due to this so huge number of parameters, very often the values assigned by the capacity to the subset of criteria are not asked directly to the DM and, consequently, several methodologies have been proposed to determine a capacity compatible with the preference information provided by the DM. For example, in [34] four different approaches are presented to deal with this problem and there is no general suggestion about which one to adopt. In this conditions, it seems very wise to take into account not one among the many capacities compatible with the DM's preference information, but, instead, the whole set of capacities compatible with the available preference information. This is the aim of the NAROR that is based on the concepts of necessary and possible preferences, holding between two alternatives a and b if a is at least as good as b for all or for at least one capacity compatible with the preferences provided by the DM.

In addition to the determination of the capacity, another important issue regarding the application of the Choquet integral is the building of a common scale on which evaluations of alternatives on considered criteria can be compared. Observe that also in this respect it is not reasonable to ask directly to the DM the values that have to be assigned to the evaluations of criteria at hand. Indeed, the DM is not able to take into consideration at the same time all the evaluations that criteria give to all alternatives and to put everything in a single scale. For this reason, we remember the famous article of Miller "The Magical Number Seven, Plus or Minus Two: Some Limits on Our Capacity for Processing Information" [122] in which it is argued that the average human brain can handle a number of objects equal to 7 ± 2. More precisely, Miller showed that this is due to the limits of one-dimensional absolute judgment and to the limits of short-term memory. This suggests to consider pairwise comparisons among objects in consequence of their manageability for the human mind, and to use some proper methodology permitting to represent pairwise comparisons by numerical evaluations of objects at hand. Some experiments showed that the use of pairwise comparisons and, even better, verbal pairwise comparisons, is much more accurate than direct estimation [123]. Consequently, to construct the common scale we propose to adopt AHP which is the most well known methodology used to build priority vectors on a homogeneous scale on the basis of pairwise comparisons of alternatives with respect to considered criteria. Some limits related to the capacity of identifying possible sources of inconsistency arise in case the number of objects to be compared is larger than seven [149]. Consequently, taking also into account the cognitive burden required to compare pairwise all the alternatives with respect to all considered criteria, we propose to focus the attention of the DM on a selected number of meaningful reference levels on each criterion. The DM is asked to supply pairwise comparisons of reference evaluations, so that they can be put on a single scale through AHP. All other evaluations are put on the common scale through linear interpolation. With this approach we reduce the effort asked to the DM for supplying the preference information and we increase the reliability of the results by taking into account pairwise comparisons given in a much more careful way (because the number of required comparisons is much smaller and the reference levels to be compared are meaningful for the DM that is supposed to participate to their selection).

Recently, some contributions proposed to conjugate AHP and the Choquet integral [15, 16, 177]. Our approach can be compared with these three works as follows:

1. [16] and [177] use AHP within a procedure to determine the capacity. More precisely, [16] uses AHP to determine a priority vector of the criteria, taking into account a capacity representing inconsistencies in the pairwise comparison matrix. Instead, [177] uses AHP to evaluate the importance of each criterion in terms of the Shapley index and, after asking the DM to supply the interactions degree of each couples of criteria, uses a nonlinear programming to get the capacity. With respect to these contributions, the method we are proposing presents a radical difference because we use NAROR, and not AHP, to define the whole set of capacities compatible with the DM's preference information containing pairwise comparisons of some real alternatives for which the DM is confident in expressing her own convictions;

2. [15] uses a pairwise comparison method to put on a common scale the evaluations of alternatives with respect to considered criteria (in fact it uses MACBETH, but AHP can be used as well) and determines the capacity on the basis of the comparisons supplied by the DM of some fictitious situations where the criteria have evaluations either totally satisfactory or totally unsatisfactory. This approach is closer to our approach but:

   - it uses a pairwise comparison method (which can be AHP as well as MACBETH) to construct evaluations of alternatives on considered criteria on the basis of pairwise comparisons of all the alternatives which can be very numerous and requiring, therefore, a strong cognitive effort to the DM; instead, in our method we consider comparisons of few reference levels for each criterion and from the evaluations of these reference levels, by linear interpolation, we assign evaluations being on a common scale to the alternatives at hand with respect to considered criteria;

   - it determines the capacity considering very specific fictitious alternatives while we can consider any alternative, so that the DM can choose those ones that she knows best and for which she feels more comfortable in expressing her preferences. Moreover, it determines only one capacity while we consider the whole set of capacities compatible with the preference information provided by the DM.

The paper is organized as follows. In the next section, we state the problem and we give the basic intuition of our proposal. Section 3 recalls the basic concepts of the Choquet integral and NAROR and it gives fundamental notions of AHP. It also introduces our methodology to build the common scale required by the Choquet integral by means of the AHP. Section 4 illustrates how our approach can be applied in a multiple criteria decision problem. Section 5 collects conclusions and further directions of research.

3.2.2 Problem statement and intuition

To explain the idea behind the Choquet integral, inspired by [68], we consider and discuss in detail a problem of evaluation of students. The Dean of a high school has to evaluate students with respect to three subjects: Mathematics (Math), Physics (Phys) and Literature (Lit). He starts using a simple weighted sum whose weights represent the importance of the different subjects. Supposing that Mathematics and Physics are more important than Literature, the weights could be 3, 3 and 2, i.e., after normalization, $w_{Math} = w_{Phys} = \frac{3}{8} = 0.375$, $w_{Lit} = \frac{2}{8} = 0.25$, respectively. Let us consider students A, B, C and D having their scores given on a 0-20 scale as shown in Table 3.17. The next to the last column of Table 3.17 gives the weighted sum of the four students at hand considering the weights of criteria previously defined.

Table 3.17: Evaluations of the students on the three considered criteria

Student/Subjects	Mathematics (Math)	Physics (Phys)	Literature (Lit)	Weighted Sum	Choquet integral
A	18	16	14	16.25	15.9
B	18	14	16	16	16.7
C	14	16	14	14.75	14.9
D	14	14	16	14.5	14.6

Applying the weighted sum, student A is preferred to student B and student C to student D. The Dean agrees with the preference of C over D, but he thinks that the preference of A over B can be questionable. Indeed student A has relatively good scores both in Mathematics and Physics, while he has a weakness in Literature. Mathematics and Physics are both important subjects, but students with a good score in Mathematics have generally also a good score in Physics and, consequently, there is an overvaluation of these students, in this case of student A. On the other hand, students good in Mathematics (or Physics) are generally not good in Literature; therefore, the Dean wants to give a bonus to student B because he has good scores in Mathematics and Literature, even if he has a weakness in Physics. The Dean wonders if it is possible to represent his preferences by changing the weights assigned to the considered subjects. Unfortunately, this is not possible because the weighted sum cannot represent the preferences of the Dean with respect to the four students. Indeed, on one hand, the preference of B over A should imply that

$$18w_{Math} + 16w_{Phys} + 14w_{Lit} < 18w_{Math} + 14w_{Phys} + 16w_{Lit}, \tag{3.5}$$

while, on the other hand, the preference of C over D should imply that

$$14w_{Math} + 16w_{Phys} + 14w_{Lit} > 14w_{Math} + 14w_{Phys} + 16w_{Lit}. \tag{3.6}$$

By (3.5) we get

$$16w_{Phys} + 14w_{Lit} < 14w_{Phys} + 16w_{Lit}, \tag{3.7}$$

while, by (3.6) we get

$$16w_{Phys} + 14w_{Lit} > 14w_{Phys} + 16w_{Lit} \tag{3.8}$$

and, clearly, (3.7) and (3.8) are incompatible. The Dean considers also the possibility to evaluate students using an additive value function such as

$$u_{Math}(score_{Math}) + u_{Phys}(score_{Phys}) + u_{Lit}(score_{Lit}), \tag{3.9}$$

with u_{Math}, u_{Phys} and u_{Lit} being non-decreasing in their arguments. Also additive value functions cannot represent the preferences of the Dean with respect to the four students A, B, C and D. In fact, the preference of B over A should imply that

$$u_{Math}(18) + u_{Phys}(16) + u_{Lit}(14) < u_{Math}(18) + u_{Phys}(14) + u_{Lit}(16), \tag{3.10}$$

while, the preference of C over D should imply that

$$u_{Math}(14) + u_{Phys}(16) + u_{Lit}(14) > u_{Math}(14) + u_{Phys}(14) + u_{Lit}(16). \tag{3.11}$$

By (3.10) we get

$$u_{Phys}(16) + u_{Lit}(14) < u_{Phys}(14) + u_{Lit}(16), \tag{3.12}$$

while, by (3.11) we get

$$u_{Phys}(16) + u_{Lit}(14) > u_{Phys}(14) + u_{Lit}(16), \tag{3.13}$$

and, again, (3.12) and (3.13) are incompatible. In fact, the preferences of the Dean do not respect *preference independence* [109] that would require that for alternatives $a, b, c, d \in A$ if

- $g_i(a) = g_i(b)$ and $g_i(c) = g_i(d)$ for $g_i \in G' \subset G$,
- $g_i(a) = g_i(c)$ and $g_i(b) = g_i(d)$ for $g_i \in G \setminus G'$,
- a is preferred to b,

then also c should be preferred to d. In simple words, preference independence would require that if two alternatives a and b have the same evaluation on a subset of criteria $G' \subset G$, then the preference of the DM should depend only on the evaluations with respect to remaining criteria, that is criteria in $G \setminus G'$, regardless the evaluation on criteria from G'. This would imply that if any two other alternatives c and d have the same evaluations on criteria from G' (even if different from the evaluations got by a and b on the same criteria) and on criteria from $G \setminus G'$ c has the same evaluations of a and d has the same evaluations of b, then c must be preferred to d. When this is not the case, the level of the evaluations on criteria G' is relevant for the preference of an alternative over the other, even if these evaluations are the same for the two compared alternatives. In these cases, preference independence is violated. Observe that this situation applies to students A, B, C and D. Indeed, A and B have the same score in Mathematics and therefore one could imagine that the Dean's preference of B over A should depend only on the scores on Physics and Literature. But if this would be true, then the Dean should prefer also D to C, because in Physics and Literature also C and D have the same scores of A and B, respectively, while they have the same score in Mathematics. However, the common score in Mathematics of C and D (14) is different from the common score in the same subject of A and B (18). Consequently C and D have a weakness in Mathematics, while A and B have a quite good score in the same subject. Thus, it is absolutely reasonable that the perspective of the Dean changes in evaluating C and D on one hand and A and B on the other hand. Comparing students C and D, since their scores in Mathematics is not so good, taking into account the technical orientation of the school, the Dean prefers C over D for her relatively good score in Physics. Instead, comparing students A and B, since their score in Mathematics is good, taking into account a favor for well equilibrated students in Science (Mathematics and Physics) and Humanities (Literature), the Dean prefers B over A for his good score in Literature. In conclusion, the preferences of the Dean violate preference independence and there are sound reasons for this. Reflecting on the problem, the Dean arrives at the conclusion that one should consider a weight not only for each subject, but also for each subset of subjects in order to represent:

- the redundancy between Mathematics and Physics; in this case the weight given to Mathematics and Physics together should be smaller than the sum of the weights given to Mathematics and Physics considered alone;
- the synergy between Mathematics and Literature or Physics and Literature; in this case the weight given to Mathematics and Literature together should be greater than the sum of the weights given to Mathematics and Literature considered alone; analogous behavior should have the weights of Physics and Literature.

Therefore, denoting by $\mu(S)$ the weight of the subset of subjects S, we could consider the following weights:

$$\mu(\{Math\}) = \mu(\{Phys\}) = 0.45, \;\; \mu(\{Lit\}) = 0.3,$$

$$\mu(\{Math, Phys\}) = 0.5, \;\; \mu(\{Math, Lit\}) = \mu(\{Phys, Lit\}) = 0.9$$

and, of course,

$$\mu(\{Math, Phys, Lit\}) = 1.$$

It can be observed that:

- according to the redundancy between Mathematics and Physics,

$$\mu(\{Math\}) + \mu(\{Phys\}) > \mu(\{Math, Phys\}),$$

- according to the synergy between Mathematics and Literature,

$$\mu(\{Math\}) + \mu(\{Lit\}) < \mu(\{Math, Lit\}),$$

- according to the synergy between Physics and Literature,

$$\mu(\{Phys\}) + \mu(\{Lit\}) < \mu(\{Phys, Lit\}).$$

Now the problem is: how to extend the weighted sum in case of interacting criteria? In other words: how to redefine the weighted sum in order to take into account not only weights for each subject (criterion), but also for each subset of subjects? Let us introduce the Choquet integral [31] explaining why it can be considered as an extension of the weighted sum in case of interacting criteria. Indeed, in this case we have to consider not only a weight for each one of the considered criteria, but also a weight for each subset of considered criteria. Let us consider the case of a set of n non-negative values $x_1, \ldots, x_n$ for which a weighted sum has to be computed on the basis of the weights $w_1, \ldots, w_n$, $w_i \geq 0$ and $w_1 + \ldots + w_n = 1$, where w_i represents the importance of the value (criterion) x_i. The weighted sum is given by

$$WS(x_1, \ldots, x_n; w_1, \ldots, w_n) = w_1 x_1 + \ldots + w_n x_n. \tag{3.14}$$

Observe that we can re-write the weighted sum as follows

$$WS(x_1, \ldots, x_n; w_1, \ldots, w_n) = \sum_{i=1}^{n} \left[\left(x_{(i)} - x_{(i-1)} \right) \sum_{j=i}^{n} w_{(j)} \right] \tag{3.15}$$

where $(\cdot)$ is a permutation of the indices $1, \ldots, n$ such that $x_{(1)} \leq \ldots \leq x_{(n)}$ and $x_{(0)} = 0$. For example, considering student A, according to (3.14) we have

$$WS(18, 16, 14; 0.375, 0.375, 0.25) = 0.375 \cdot 18 + 0.375 \cdot 16 + 0.25 \cdot 14 = 16.25$$

while, according to (3.15) we have

$$WS(18, 16, 14; 0.375, 0.375, 0.25) =$$

$$= (14 - 0) \cdot (0.375 + 0.375 + 0.25) + (16 - 14) \cdot (0.375 + 0.375) + (18 - 16) \cdot 0.375 = 16.25.$$

In case of absence of interaction between criteria, the importance of a subset of criteria is

$$\mu(S) = \sum_{i \in S} w_i,$$

so that

$$\sum_{j=i}^{n} w_{(j)} = \mu(\{(i), \ldots, (n)\})$$

and (3.15) can be written as follows:

$$WS(x_1, \ldots, x_n; w_1, \ldots, w_n) = \sum_{i=1}^{n} \left[\left(x_{(i)} - x_{(i-1)} \right) \mu(\{(i), \ldots, (n)\}) \right]. \tag{3.16}$$

Now the Choquet integral, denoted by $C_\mu(x_1, \ldots, x_n)$, is formulated exactly as (3.16), i.e.

$$C_\mu(x_1, \ldots, x_n) = \sum_{i=1}^{n} \left[\left(x_{(i)} - x_{(i-1)} \right) \mu(\{(i), \ldots, (n)\}) \right] \tag{3.17}$$

and this formulation holds also in case of interacting criteria.

Therefore, according to (3.17), considering the capacity defined above, the Choquet integral of the scores of student A is

$$C_\mu(18, 16, 14) = (14 - 0) \cdot \mu(\{M, P, L\}) + (16 - 14) \cdot \mu(\{M, P\}) + (18 - 16) \cdot \mu(\{M\}) =$$

$$= (14 - 0) \cdot 1 + (16 - 14) \cdot 0.5 + (18 - 16) \cdot 0.45 = 15.9.$$

The Choquet integral of the scores of students B, C and D can be computed analogously obtaining the results shown in the last column of Table 3.17. One can see that, in this case, student B is evaluated better than student A and student C is evaluated better than student D.

We have shown that the Choquet integral is able to take into account interactions between criteria. However, the application of the Choquet integral presents two relevant problems that did not appear clearly in the above didactic example:

a) Differently from the usual weighted sum, where we have to assign only one weight to each criterion, the Choquet integral requires to assign a weight to each subset of criteria. This problem becomes very relevant when the number of criteria is high. Indeed, for n criteria we have to assign n values in case of the weighted sum (one for each criterion), while we have to assign 2^n weights (one for each subset of criteria) in case of the Choquet integral (in fact $2^n - 2$ weights because the weight assigned to the empty set is null and the weight assigned to the whole set of criteria is equal to one). For example, if we have 3 criteria, we have to assign $2^3 = 8$ weights which become $2^4 = 16$ if we have 4 criteria and so on. It is clear that asking to the DM to provide a large number of weights is not reasonable. It is also worthwhile to observe that the interpretation of these weights is not trivial for the DM because of the interaction of several criteria at once.

b) As it can be seen from (3.15), the Choquet integral requires that the evaluations with respect to all considered criteria are on the same scale. In fact, for each alternative, the evaluations on the considered criteria need to be ordered from the smallest to the greatest. In the example of Table 3.17, if we consider student A, it is immediate to conclude that a score of 18 in Mathematics is greater than a score of 16 in Physics which, in turn, is greater than a score of 10 in Literature. Even more, as shown by (3.17), to compute the Choquet integral we have also to consider the difference between evaluations of the same alternative on different criteria. Therefore, for example, taking into consideration student A, we must be able to say that the difference in the score between Mathematics and Physics is 18-16=2, and that it is meaningful to state that it is one half of the difference between Mathematics and Literature which is 18-14=4. However, let suppose we have to rank a set of cars to decide which one to buy and suppose we want to use the Choquet integral to evaluate, for example, Audi A3 (3 doors) having the following characteristics:

   Price: €22,140
   Acceleration: 10.3 second to arrive from 0 to 100 km/h
   Maximum speed: 193 km/h
   Fuel consumption: 4.9 l/km.

   How to order these evaluations for computing the Choquet integral? In other words, is a price of €22,140 more valuable than a maximum speed of 193 km/h? And, even much more problematic: how can we give a value to this difference?

Some answers have been given in the literature to the two above points. Regarding point a), the most convincing answer seems the proposal of inducing the weights from some indirect preference information provided by the DM in terms of pairwise comparisons of some reference alternatives and in terms of relative importance and interaction between criteria [119]. Recently, in the same direction, NAROR [8] has been proposed. It permits to consider the whole set of compatible weights, i.e. the whole set of weights satisfying the preference information provided by the DM, by the use of a necessary and a possible preference relation as in any Robust Ordinal Regression method (ROR; [35, 36, 86]). The necessary and possible preference relations hold between alternatives a and b if a is not worse than b for all, or for at least one, of the sets of weights compatible with the DMs preferences, respectively (for a discussion on the axiomatic basis of necessary and possible preference relations see [64]).

With respect to point b), very often a normalization of evaluations on each criterion is done considering an “unacceptable” minimal value and an “ideal” maximal value for each criterion and considering a linear interpolation between these two extremes. A more sophisticated methodology permitting to build one scale and one capacity for the Choquet integral on the basis of the preference information provided by the DM has been proposed in [6] and it has been further developed in [4]. However, these two approaches are heuristics and not exact algorithms.

In this paper we propose to deal with problems a) and b) in a systematic way as follows:

- First we construct a common scale for all criteria using the AHP. The advantage of using AHP in this context is given by the possibility of building the scale using the preferences provided by the DM. Moreover, for the considered criteria, the evaluations on the scale obtained by AHP are comparable between them and therefore we can apply the Choquet integral; it is worthwhile to observe that our use of AHP is parsimonious with respect to the information asked to the DM, in the sense that with respect to each considered criterion, we shall not ask the DM to compare pairwise all the alternatives, as it is commonly done, but we shall ask the DM to compare some reference levels on the considered criteria. The other non reference evaluations are obtained by interpolating the values assigned by AHP to the reference levels. We believe that this is another important contribution of our work which goes beyond the mere use of AHP in a MCDA procedure based on the Choquet integral preference model. In fact, it can be used in any decision problem where AHP has to be applied to a large set of alternatives. In this way, the DM avoids to answer to a long and tiring list of questions related to all pairwise comparisons of the alternatives at hand.

- Using the evaluations expressed in the scale obtained by means of the AHP method, we proceed with the application of the NAROR. In this way, we consider all the set of weights which are compatible with the preference information provided by the DM avoiding to consider only one set of weights chosen in an arbitrary way in the whole family of compatible sets. Since the necessary and possible preference relations obtained by NAROR can present some difficulty to be handled by the DM, we present her also a complete ranking of the considered alternatives obtained by computing the so called most representative value function [7, 53]. This is a value function corresponding to the Choquet integral with respect to the capacity giving the “best” representation not only of all the preferences supplied by the DM, but also of the necessary and the possible preference relations.

We believe that the proposed combination of AHP and NAROR represents the most convenient way to deal with the two discussed problems of determining in a reasonable and meaningful way a set of weights and a common scale for the considered criteria when the Choquet integral is used to represent interaction between criteria.

3.2.3 Methodology

IntroductionThis section presents the methods used to solve the problem described in Section 3.2.2. First, we formalize the Choquet integral preference model for MCDA problems (Section 2.2.2). As the capacities to be used in the Choquet integral are difficult to elicit, we use the NAROR (Section 3.2.3), to indirectly infer all capacities compatible with the information provided by the DM. In Section 3.2.3 basic principles of AHP are then recalled, describing after how to use it for building the common scale, which is needed to apply the Choquet integral. In Section 3.2.3 we show how the number of pairwise comparisons asked by the AHP can be decreased by taking into account only reference points and then interpolating the results.

The Choquet integral preference modelA set function $\mu : 2^G \rightarrow [0, 1]$ is called a capacity (fuzzy measure) on 2^G (being the power set of G, i.e. the set of all subsets of G) if the following properties hold:

1a) $\mu(\emptyset) = 0$ and $\mu(G) = 1$ (boundary conditions),

2a) $\forall\, S \subseteq T \subseteq G,\ \mu(S) \leq \mu(T)$ (monotonicity condition).

For any $T \subseteq G$, $\mu(T)$ represents the total weight of criteria from T, which is not supposed to be additive, i.e. it is not necessarily true that for any $S, T \subseteq G$ such that $S \cap T = \emptyset$, one has $\mu(S \cup T) = \mu(S) + \mu(T)$.
In this case, we have to define $2^{|G|} - 2$ non additive weights $\mu(S)$, $\emptyset \subset S \subset G$, since the values $\mu(\emptyset) = 0$ and $\mu(G) = 1$ are already known.
Given $a \in A$ and a capacity μ on 2^G, the Choquet integral [31], as above explained, gives the analogous of the weighted sum in case of additive weights, and it is defined as follows:

$$C_\mu(a) = \sum_{i=1}^{n} \left[\left(g_{(i)}(a) - g_{(i-1)}(a) \right) \mu\left(\{(i), \ldots, (n)\} \right) \right]$$

where $(\cdot)$ reorders the criteria so that $g_{(1)}(a) \leq \ldots \leq g_{(n)}(a)$ and $g_{(0)}(a) = 0$.
It is useful to consider also the Möbius representation of a capacity μ being the function $m : 2^G \rightarrow \mathbb{R}$ [140, 153] such that, for all $S \subseteq G$

$$\mu(S) = \sum_{T \subseteq S} m(T).$$

The Möbius representation $m(S)$ can be obtained from $\mu(S)$ as follows:

$$m(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \mu(T).$$

For the Möbius representation [30], properties 1a) and 2a) become

1b) $m(\emptyset) = 0, \ \sum_{T \subseteq G} m(T) = 1,$

2b) $\forall\, i \in G$ and $\forall R \subseteq G \setminus \{i\}\,,\ m(\{i\}) + \sum_{T \subseteq R} m(T \cup \{i\}) \geq 0.$

The Möbius representation is important in applications, because it permits to express the Choquet integral in a linear form (but in a space different from that one of values given by criteria from G), formulating the Choquet integral as a weighted sum of minimum values given to the considered alternative $a \in A$ by all subsets of criteria T from G [66],

$$C_\mu(a) = \sum_{T \subseteq G} m(T) \min_{i \in T} g_i\,(a)\,.$$

However, even if expressed in linear form with the above formula, the use of the Choquet integral preference model remains difficult because we have to determine the $2^{|G|}$ values $m(T)$, $T \subseteq G$. In order to reduce the number of parameters to be determined and to get a simpler formulation for the Choquet integral, the concept of k-additive capacity, $k = 1, \ldots, n$, has been introduced [69]. Formally a capacity is k-additive if $m(T) = 0$ for $T \subseteq G$ such that $|T| > k$. Intuitively a capacity is k-additive if it considers interactions between no more than k criteria. In MCDA, 2-additive capacities are often considered because it is reasonable to expect that the DM could supply preference information on positive and negative interactions between couples of criteria, while it seems that interactions between three, four and more criteria are more difficult, or even sometimes impossible, to evaluate. It is to observe that a 2-additive capacity μ in terms of Möbius representation has the following formulation

$$\mu(S) = \sum_{i \in S} m\left(\{i\}\right) + \sum_{\{i,j\} \subseteq S} m\left(\{i,j\}\right),\ \forall S \subseteq G.$$

Thus, from the computational point of view, 2-additive capacities require to induce the value of only

$n+\binom{n}{2}$ parameters, being a value $m(\{i\})$ for every criterion i and a value $m(\{i,j\})$ for every couple of criteria $\{i,j\}$. For 2-additive capacities, properties 1b) and 2b) have to be reformulated as follows:

1c) $m(\emptyset)=0,\ \sum_{i\in G} m(\{i\})+\sum_{\{i,j\}\subseteq G} m(\{i,j\})=1,$

2c) $\begin{cases} m(\{i\})\geq 0,\ \forall i\in G, \\ m(\{i\})+\sum_{j\in T} m(\{i,j\})\geq 0,\ \forall i\in G \text{ and } \forall\, T\subseteq G\setminus\{i\},\, T\neq\emptyset. \end{cases}$

When the Choquet integral is adopted in MCDA, the importance of a criterion $i\in G$ is not evaluated considering only the value assigned by the capacity to the criterion i alone, i.e. $\mu(\{i\})$, but also taking into consideration all its interactions, i.e., in case of 2-additive capacities, considering $\mu(\{i,j\})$ for all $j\in G\setminus\{i\}$. So doing, the importance of criterion $i\in G$ is expressed by the Shapley value [154] that, in case of a 2-additive capacity, has the following formulation:

$$\varphi(\{i\})=m(\{i\})+\sum_{j\in G\setminus\{i\}}\frac{m(\{i,j\})}{2}.$$

As pointed out above, with respect to a criterion $i\in G$, in general the Shapley index $\varphi(\{i\})$ is different from the weight $\mu(\{i\})$ assigned to the criterion i by the capacity μ. Our methodology takes into account this fact by modeling DM's preference information related to comparison between the importance of criteria in terms of Shapley index $\varphi(\{i\})$. In this way, we acknowledge that the importance of criterion i does not depend on itself only but also on its interactions with the other criteria at hand.

Among the preference information that the DM can supply there is also the sign and the magnitude of the interaction $\varphi(\{i,j\})$ of couples of criteria $\{i,j\}\subseteq G$ [128]. For a 2-additive capacity μ the interaction is given by the Möbius representation of the couple $\{i,j\}$, i.e.

$$\varphi(\{i,j\})=m(\{i,j\}).$$

Robust Ordinal Regression and NAROR

Intuition of NAROR Consider the example proposed in Section 3.2.2. Suppose now that the Dean wants to evaluate three new students E,F and H, whose scores are shown in Table 3.18.

The Dean wants to apply the Choquet integral but he wants to be sure about his evaluations, and, consequently, he wants to consider all the capacities μ that are coherent with his preferences. Therefore he takes into account the following constraints for the values taken by μ:

Table 3.18: Evaluations of three new students on the three considered criteria

Student/Subjects	Mathematics (M)	Physics (P)	Literature (L)	Choquet integral with respect to capacity μ_1	Choquet integral with respect to capacity μ_2
E	19	14	15	16.48	16.3125
F	18	18	14	16	16.5
H	18	14	18	16.4	17.25

- $\mu(\{Math, Lit\}) > \mu(\{Math, Phys\})$;

  Indeed, by considering the preference of B over A translated by the inequality

$$C_\mu(18, 14, 16) > C_\mu(18, 16, 14),$$

and applying (3.17), we have that

$$14\mu(\{Math, Phys, Lit\}) + (16 - 14)\mu(\{Math, Lit\}) + (18 - 16)\mu(\{Math\})$$

$$>$$

$$14\mu(\{Math, Phys, Lit\}) + (16 - 14)\mu(\{Math, Phys\}) + (18 - 16)\mu(\{Math\})$$

and, consequently,

$$\mu(\{Math, Lit\}) > \mu(\{Math, Phys\}). \tag{3.18}$$

- $\mu(\{Phys\}) > \mu(\{Lit\})$;

  Indeed, by considering the preference of C over D translated by the inequality,

$$C_\mu(14, 16, 14) > C_\mu(14, 14, 16)$$

and applying (3.17) we have that

$$14\mu(\{Math, Phys, Lit\}) + (16 - 14)\mu(\{Phys\})$$

$$>$$

$$14\mu(\{Math, Phys, Lit\}) + (16 - 14)\mu(\{Lit\})$$

and, consequently,

$$\mu(\{Phys\}) > \mu(\{Lit\}). \tag{3.19}$$

- $$\mu(\{Math\}) = \mu(\{Phys\}) \text{ and } \mu(\{Math, Lit\}) = \mu(\{Phys, Lit\}) \quad (3.20)$$

  because Mathematics and Physics are considered equally important by the Dean.

Considering all the capacities μ satisfying constraints (3.18)-(3.20), the Dean arrives at the following conclusions:

- student H is preferred to student F for every capacity μ compatible with his preferences; indeed

  $$C_\mu(18, 14, 18) > C_\mu(18, 18, 14)$$

  that, by (3.17), becomes

  $$14\mu(\{Math, Phys, Lit\}) + (18 - 14)\mu(\{Math, Lit\})$$

  $$>$$

  $$14\mu(\{Math, Phys, Lit\}) + (18 - 14)\mu(\{Math, Phys\})$$

  which is always true by eq. (3.18);

- among the capacities μ compatible with his preferences, there are some for which student E is preferred to student F and there are others for which student F is preferred to student E; for example, student E is preferred to student F for the capacity μ_1 such that

  $$\mu_1(\{Math, Lit\}) = \mu_1(\{Phys, Lit\}) = 0.6, \ \mu_1(\{Math, Phys\}) = 0.5,$$

  $$\mu_1(\{Math\}) = \mu_1(\{Phys\}) = 0.47, \ \mu_1(\{Lit\}) = 0.1, \ \mu_1(\{Math, Phys, Lit\}) = 1,$$

  while, student F is preferred to student E for the capacity μ_2 such that

  $$\mu_2(\{Math, Lit\}) = \mu_2(\{Phys, Lit\}) = 0.8125, \ \mu_2(\{Math, Phys\}) = 0.625,$$

  $$\mu_2(\{Math\}) = \mu_2(\{Phys\}) = 0.375, \ \mu_2(\{Lit\}) = 0.1875, \ \mu_2(\{Math, Phys, Lit\}) = 1;$$

  the evaluations given to students E, F and H by the Choquet integral with respect to capacities μ_1 and μ_2 are shown in the last two columns of Table 3.18;

- among the capacities μ compatible with his preferences, there are some for which student E is preferred to student H and there are others for which student H is preferred to student E; for example, student E is preferred to student H for the capacity μ_1 while, student H is preferred to student E for the capacity μ_2.

In conclusion, the Dean is convinced that there is no doubt about the preference of student H over student F. In this case, we speak of necessary preference. However, there are some doubts about the preference between student E and student F, and between student E and student H. In this case we speak of possible preferences. The following subsection 3.2.3 recalls basic concepts of NAROR [8] that permits to define systematically necessary and possible preferences when using the Choquet integral.

The formal model of NAROR

As observed in the example of the previous Section, in general, there is more than one capacity that permits to represent the preference information provided by the DM through the Choquet integral. Since choosing only one of these compatible capacities is always arbitrary to some extent, following the principles of ROR [35, 36, 86], we take into account all the capacities compatible with the preference information provided by the DM through NAROR [8] that we shall recall in the following. To get the values $\mu(T)$ that the capacity μ assigns to all the subsets T of G, one can use a direct or an indirect technique. The direct technique asks the values $\mu(T)$ or the corresponding Möbius representation directly to the DM while the indirect technique infers the values $\mu(T)$ from some preference information provided by the DM ([6, 119]; for an extensive review on the topic see also [70] and [74]).

When using an indirect technique, the DM can supply the following information with respect to a subset of alternatives $A^R \subseteq A$:

- a partial preorder $\succsim$ on A^R whose meaning is: for $a^*, b^* \in A^R$

$$a^* \succsim b^* \Leftrightarrow \text{“}a^* \text{ is at least as good as } b^*\text{”};$$

- a partial preorder $\succsim^*$ on $A^R \times A^R$, whose meaning is: for a^*, b^*, c^*, $d^* \in A^R$,

$$(a^*, b^*) \succsim^* (c^*, d^*) \Leftrightarrow \text{“}a^* \text{ is preferred to } b^* \text{ at least as much as } c^* \text{ is preferred to } d^*\text{”};$$

- a partial preorder $\succsim_1$ on G, whose meaning is: for $g_i, g_j \in G$

$$g_i \succsim_1 g_j \Leftrightarrow \text{“criterion } g_i \text{ is at least as important as criterion } g_j\text{”};$$

- a partial preorder $\succsim_1^*$ on $G \times G$, whose meaning is: for g_i, g_j, g_k, $g_l \in G$,

$(g_i, g_j) \succsim_1^* (g_k, g_l) \Leftrightarrow$ “the difference of importance between criteria g_i and g_j is no lower than the difference of importance between criteria g_k and g_l”;

- the sign of the interaction between criteria g_i and g_j, with $g_i, g_j \in G$:
  (a) g_i and g_j are positively interacting,
  (b) g_i and g_j are negatively interacting.

In the following, as usual, $\sim$ (indifference) and $\succ$ (preference) so as $\sim^*$ and $\succ^*$ denote the symmetric and the asymmetric part of $\succsim$ and $\succsim^*$, respectively, that is,

- $a^* \sim b^*$ is equivalent to $a^* \succsim b^*$ and $b^* \succsim a^*$, while
- $a^* \succ b^*$ is equivalent to $a^* \succsim b^*$ and not$(b^* \succsim a^*)$,

as well as

- $(a^*, b^*) \sim^* (c^*, d^*)$ is equivalent to $(a^*, b^*) \succsim^* (c^*, d^*)$ and $(c^*, d^*) \succsim^* (a^*, b^*)$, while
- $(a^*, b^*) \succ^* (c^*, d^*)$ is equivalent to $(a^*, b^*) \succsim^* (c^*, d^*)$ and not$[(c^*, d^*) \succsim^* (a^*, b^*)]$.

The preference information provided by the DM permits to define the following set E^{A^R} of constraints representing the set of all the capacities compatible with the preference information given by the DM:

$$\left.\begin{array}{ll}
C_\mu(a^*) \geq C_\mu(b^*) \ \text{ if } \ a^* \succsim b^*, & \\
C_\mu(a^*) \geq C_\mu(b^*) + \varepsilon \ \text{ if } \ a^* \succ b^*, & \\
C_\mu(a^*) = C_\mu(b^*) \ \text{ if } \ a^* \sim b^*, & \\
C_\mu(a^*) - C_\mu(b^*) \geq C_\mu(c^*) - C_\mu(d^*) + \varepsilon \ \text{ if } \ (a^*, b^*) \succ^* (c^*, d^*), & \\
C_\mu(a^*) - C_\mu(b^*) = C_\mu(c^*) - C_\mu(d^*) \ \text{ if } \ (a^*, b^*) \sim^* (c^*, d^*), & \\
\varphi(\{i\}) \geq \varphi(\{j\}) \ \text{ if } \ i \succsim_1 j, & \\
\varphi(\{i\}) = \varphi(\{j\}) \ \text{ if } \ i \sim_1 j, & \\
\varphi(\{i,j\}) \geq \varepsilon \ \text{ if criteria } i \text{ and } j \text{ are positively interacting with } i,j \in G, & \\
\varphi(\{i,j\}) \leq -\varepsilon \ \text{ if criteria } i \text{ and } j \text{ are negatively interacting with } i,j \in G, & \\
m\left(\{\emptyset\}\right) = 0, \ \sum\limits_{i \in G} m\left(\{i\}\right) + \sum\limits_{\{i,j\} \subseteq G} m\left(\{i,j\}\right) = 1, & [NC] \\
m\left(\{i\}\right) \geq 0, \ \forall i \in G, & [MC_1] \\
m\left(\{i\}\right) + \sum\limits_{j \in T} m\left(\{i,j\}\right) \geq 0, \forall i \in G \text{ and } \forall\, T \subseteq G \setminus \{i\}, T \neq \emptyset, & [MC_2]
\end{array}\right\}$$

where ε is an auxiliary variable used to transform the strict inequality constraints in weak inequality constraints. If $\varepsilon^* > 0$, where $\varepsilon^* = \max \varepsilon$ subject to E^{A^R}, then there exists at least one capacity compatible with the preference information provided by the DM. If there is not any capacity compatible with the preference information provided by the DM, one can use techniques described in [125] to determine the minimal set of pieces of preference information that could be revised by the DM in order to remove the incompatibility of constraints in E^{A^R}.

Note 3.2.1. *From the computational point of view, the previous problem is a linear programming problem composed of* $|G| + \binom{|G|}{2} + 1$ *variables and, in particular, one Möbius parameter* $m(\{i\})$ *for each criterion* $i \in G$*, one Möbius parameter* $m(\{i,j\})$ *for each pair of criteria* $\{i,j\} \subseteq G$[2] *and the variable* ε*. In the following, to simplify the notation, we shall denote by* (m, ε) *the vector composed of the variables and we shall call it "the model".*
The set $[NC]$ *is composed of two equality constraints; the set* $[MC_1]$ *is composed of* $|G|$ *inequality constraints (one for each criterion* $i \in G$*) while the set* $[MC_2]$ *is composed of* $|G| \cdot \left(2^{|G|-1} - 1\right)$ *inequality constraints (*$2^{|G|-1} - 1$ *for each criterion* $i \in G$*). Therefore, the number of monotonicity*

[2] Let us observe that $m(\{i,j\}) = m(\{j,i\})$ for each pair of criteria $\{i,j\}$ and this is the reason for which the Möbius paremeters $m(\{i,j\})$ are $\binom{|G|}{2}$.

and normalization constraints will be equal to $2+|G|\cdot 2^{|G|-1}$*. To these monotonicity and normalization constraints should be added as many equality or inequality constraints as the number of pieces of information provided by the DM. In particular, one equality constraint for each piece of indifference information (indifference between couples of alternatives or no difference between the importance of two criteria) and one inequality constraint for each piece of preference information (preference between two alternatives, intensity of preference between couples of alternatives, preference in the importance of two criteria) or for each positive or negative interaction.*

In general, there could exist more than one capacity compatible with the preference information provided by the DM and to take into account all these compatible capacities, one can compute the necessary ($\succsim^N$) and the possible ($\succsim^P$) preference relations on the set of alternatives A as described below.

Given the following sets of constraints,

$$\left.\begin{array}{l} C_\mu(b) \geq C_\mu(a) + \varepsilon \\ E^{A^R}, \end{array}\right\} E^N(a,b) \qquad \left.\begin{array}{l} C_\mu(a) \geq C_\mu(b) \\ E^{A^R}, \end{array}\right\} E^P(a,b)$$

we have that:

- $a \succsim^N b$ if $E^N(a,b)$ is infeasible or if $\varepsilon^N \leq 0$, where $\varepsilon^N = \max \varepsilon$ subject to $E^N(a,b)$,
- $a \succsim^P b$ if $E^P(a,b)$ is feasible and $\varepsilon^P > 0$, where $\varepsilon^P = \max \varepsilon$ subject to $E^P(a,b)$.

Note 3.2.2. *To get the necessary and possible preference relations one has to solve two linear programming problems in which the considered variables are the same of* E^{A^R}*, while the number of constraints augmented by one unity.*

To check if $a \succsim^N b$*, we add the constraint* $C_\mu(b) \geq C_\mu(a)+\varepsilon$*, corresponding to the strict preference of* b *over* a*, to the set* E^{A^R} *containing the constraints translating the preference information provided by the DM and the monotonicity and normalization constraints. Since we already knew that the set* E^{A^R} *is feasible and that there exists at least one model* (m,ε) *such that* $\varepsilon > 0$*, we have the following possible cases:*

- $E^N(a,b)$ *is infeasible: this means that the strict preference of* b *over* a *is not compatible with the preference information provided by the DM since there does not exist any model* (m,ε) *for which all the constraints are satisfied (independently from the sign of* ε*). Because it is not possible to find any model compatible with the preference information provided by the DM for* *which b is strictly preferred to a, then a will be at least as good as b for all compatible models. Consequently,* $a \succsim^N b$.

- $E^N(a,b)$ *is feasible but* $\varepsilon^N \leq 0$*: in this case, even if there exist some model* (m,ε) *for which the constraints are all satisfied, none of them has one* ε *greater than zero. This means that for none of these models* $C_\mu(b) > C_\mu(a)$ *and again, a is therefore at least as good as b for all compatible models, implying that* $a \succsim^N b$.

- $E^N(a,b)$ *is feasible but* $\varepsilon^N > 0$*: in this case, there exist at least one model* (m,ε) *for which all constraints are satisfied but, there exist at least one model presenting* $\varepsilon > 0$*. This means that b could be strictly preferred to a and, consequently, it is not true that a is at least as good as b for all compatible models.*

To check if $a \succsim^P b$*, we add the constraint* $C_\mu(a) \geq C_\mu(b)$*, corresponding to the statement "a is at least as good as b", to the set* E^{A^R}*. Since we already know that the set* E^{A^R} *is feasible and that there exists at least one model* (m,ε) *such that* $\varepsilon > 0$*, we have the following possible cases:*

- $E^P(a,b)$ *is infeasible: this means that the weak preference of a over b is incompatible with the preference information provided by the DM since there does not exist any model* (m,ε) *for which all the constraints are satisfied (independently from the sign of* ε*). Because it is not possible to find any model compatible with the preference information provided by the DM for which a is weakly preferred to b, then b will be strictly preferred to a for all compatible models that implies that* $a \not\succsim^N b$.

- $E^P(a,b)$ *is feasible but* $\varepsilon^P > 0$*: in this case, there exist some model* (m,ε) *presenting* $\varepsilon > 0$ *for which the constraints are all satisfied. This implies that there exists at least one compatible model for which a is at least as good as b, that is* $a \succsim^P b$.

- $E^P(a,b)$ *is feasible but* $\varepsilon^P \leq 0$*: this means that there exists some model* (m,ε) *for which all constraints are satisfied but none of these models presents an* $\varepsilon > 0$*. Therefore, the new constraint* $C_\mu(a) \geq C_\mu(b)$ *is not compatible with the preference information provided by the DM and, consequently, it is not true that* $a \succsim^P b$.

Finally, let us briefly remind that for each pair of alternatives $a, b \in A$, $a \succsim^N b$ *implies* $a \succsim^P b$ *(therefore we can state that* $\succsim^N \subseteq \succsim^P$*), and that for all* $a, b \in A$, $a \succsim^N b$ *or* $b \succsim^P a$ *(for further important properties of the necessary and possible preference relations, see [86]). We have reminded this point to underline that to compute the necessary and possible preference relations we do not* *need, in general, to solve two LP problems for each couple of alternatives (a, b). In particular, after computing the necessary preference relation for each couple of alternatives (a, b), we need to compute only the possible preference relation for the couples (b, a) such that $a \succsim^N b$ but not$(b \succsim^N a)$. Indeed, by considering the two properties mentioned above, we can have the following three cases:*

- $a \succsim^N b$ *and* $b \succsim^N a \Rightarrow a \succsim^P b$ *and* $b \succsim^P a$*;*
- $a \succsim^N b$ *and not*$(b \succsim^N a) \Rightarrow a \succsim^P b$ *but nothing could be said about* $b \succsim^P a$*;*
- *not*$(a \succsim^N b)$ *and not*$(b \succsim^N a) \Rightarrow b \succsim^P a$ *and* $a \succsim^P b$*.*

In some cases, and also in our approach, one needs to assign a real number to the overall evaluation of each alternative in order to obtain a complete ranking of the alternatives.
With this aim, based on the results of the ROR, among all the compatible models one can compute the most representative model being that one maximizing the difference in the performance of two alternatives a and b for which $a \succsim^N b$ but not$(b \succsim^N a)$, and minimizing the difference in the performances of two alternatives a and b such that $a \succsim^P b$ and $b \succsim^P a$ [7, 53].
The considered procedure is composed of two steps:

(S_1) Solving the following optimization problem:

$$\left.\begin{array}{l} \max \varepsilon \;\text{ subject to} \\ C_\mu(a) \geq C_\mu(b) \;\text{ if } a \succsim^N b \text{ and not}(b \succsim^N a), \\ E^{A^R}. \end{array}\right\} E_1$$

(S_2) Denoted by ε_1 the optimal value of epsilon obtained in the previous step, solving the following optimization problem:

$$\left.\begin{array}{l} \min \delta \;\text{ subject to} \\ \left.\begin{array}{l} C_\mu(a) - C_\mu(b) \leq \delta, \\ C_\mu(b) - C_\mu(a) \leq \delta, \end{array}\right\} \text{ if } a \succsim^P b \text{ and } b \succsim^P a, \\ E_1, \\ \varepsilon = \varepsilon_1. \end{array}\right\}$$

Analytic Hierarchy Process

Description of the methodAHP is a multi-criteria decision making (MCDM) method that helps the DM in solving a complex problem having multiple conflicting criteria [93, 144, 145]. In its full version, AHP is structured in a hierarchy, where the decision goal is located on the top. The children of the tree are the criteria and sub-criteria to be satisfied by each alternative.
In our paper, we are not solving the whole problem with AHP but we are only inducing a rating table. Therefore, only the core feature of AHP, the pairwise comparison matrix, is needed. The method of pairwise comparisons provides more accurate results than direct evaluations, primarily due to the fact that the DM is asked to concentrate only on two elements at a time [49, 92, 123]. With respect to MACBETH [9], another methodology to construct a scale from pairwise comparisons, AHP has the advantage to consider also indirect comparisons to derive priorities [94, 176], and this, on one hand, permits to better control the consistency of information supplied by the DM, and, on the other hand, gives more reliable results.

The pairwise comparisons are entered in a positive reciprocal matrix $\mathbf{A} = [a_{ij}]$ of dimension n, where n is the number of considered "objects", and a_{ij} expresses how many times "object" i is "greater" than "object" j. Indeed if p_i measures the magnitude of i and p_j the magnitude of j, we should have $a_{ij} = \frac{p_i}{p_j}$. Usually, following Saaty, a_{ij} is determined by asking the DM a verbal judgment using the scale "moderately more dominant", "strongly more dominant", "very strongly more dominant", and "extremely more dominant" which are numerically coded as 3, 5, 7, and 9, respectively, with 2, 4, 6, and 8 for compromise between the previous values. The values a_{ij} from matrix A are consistent if

1) $a_{ij} = \frac{1}{a_{ji}}$ for all i and j, and this is always satisfied because, as above mentioned, matrix $\mathbf{A}$ is supposed reciprocal, and

2) $a_{ij}a_{jk} = a_{ik}$ for all i, j and k.

It can also be written that values a_{ij} from matrix $\mathbf{A}$ are consistent if and only if there exist values $p_1, \ldots, p_n$ such that for $i = 1, \ldots, n$

$$a_{i1}p_1 + \ldots + a_{in}p_n = n \cdot p_i$$

which can be written as

$$\mathbf{A} \cdot \mathbf{p} = n \cdot \mathbf{p} \tag{3.21}$$

where $\mathbf{p} = [p_i]$ is the priorities vector. In this case the priorities $p_1, \ldots, p_n$ and, consequently, the vector $\mathbf{p}$, can be easily determined as follows:

$$p_i = \frac{a_{ij}}{a_{1j} + \ldots + a_{nj}} \quad \text{for } i = 1, \ldots, n. \tag{3.22}$$

If (3.21) holds, (3.22) gives the same values for any $j = 1, \ldots, n$.
However, condition (3.21) is rarely satisfied which implies that (3.22) cannot be used because it gives different values to p_i when different j are considered. Therefore several methods have been proposed to induce the priority vector $\mathbf{p}$ and among them the most well-known is the eigenvalue method calculating the priorities corresponding to matrix $\mathbf{A}$ as follows:

$$\mathbf{A} \cdot \mathbf{p} = \lambda_{max} \cdot \mathbf{p}$$

where $\lambda_{\max}$ is the maximal eigenvalue of matrix $\mathbf{A}$.
In order to declare the comparison matrix consistent enough for calculating credible priorities, it must pass a consistency check. Consistency Ratio (CR) is defined as:

$$CR = CI/RI,$$

where RI is the Random Index (the average CI of 500 randomly filled matrices).
CI is the Consistency Index

$$CI = (\lambda_{max} - n)/(n - 1).$$

In order to make the rating on different criteria commensurable, we normalise the priorities with respect to the maximum and minimum score on each criterion as follows:

$$p_i^* = \frac{p_i - p_{min}}{p_{max} - p_{min}}$$

where:

- p_i^* is the rating of the score i,
- p_i is the calculated priority of the score i,
- p_{max} is the priority of the maximum score,
- p_{min} is the priority of the minimum score.

Observe that the rating of p_i^* will be in the range $[0, 1]$, and $p_i^* = 1$ if $p_i = p_{max}$ and $p_i^* = 0$ if $p_i = p_{min}$.

Reducing the number of pairwise comparisonsAHP permits to build a scale on each criterion starting from ratio evaluations given by the DM. This is very important in multiple criteria decision making because numerical evaluations of alternatives on considered criteria permit to proceed towards their aggregation in a single overall evaluation allowing to compare comprehensively alternatives between them. However, the application of the AHP becomes troublesome when the number of considered alternatives is high. Indeed, if the number of alternatives is m, then there is the necessity of $\frac{m(m-1)}{2}$ pairwise comparisons for each considered criterion, such that, for n criteria we have a total of $n\frac{m(m-1)}{2}$ pairwise comparisons to ask to the DM. For example, for a not so complex problem with 7 criteria and 7 alternatives, the total number of pairwise comparisons to ask to the DM is 147, which becomes 450 in a problem with 10 criteria and 10 alternatives and 1900 in a problem with 10 criteria and 20 alternatives. Thus the use of AHP requires a consistent cognitive efforts from the DM and this can deteriorate the quality of the information provided with the risk that the MCDA methodology gives back a result which is not enough reliable. Consider also that, sometimes, there is the necessity to get a decision model that can be applied to a set of alternatives very large and not predefined at the beginning. For instance, suppose that one wants to develop a model to assess credit score for customers of a bank taking into account several financial ratios such as return on equity, current ratio, debt ratio and so on. In this case, it is necessary to build a decision model which is universally applicable to all the customers requiring a credit to the bank. Thus, we should consider thousands of potential customers and, even if we would be able to get the billions of pairwise comparisons necessary to apply AHP, in any case, the customers that could apply for a credit cannot be known in advance. Therefore, there is the necessity to develop a method permitting to apply AHP also in these cases. For this reason, we propose to fix a small number of representative points in the scale of each criterion and to ask the DM to compare pairwise these points. After obtaining through AHP a normalized evaluation for these points, the evaluations of all other alternatives with respect to the considered criteria can be obtained by linear interpolation. More formally, supposing for the sake of simplicity that all criteria have a numerical scale that is monotonic increasing with respect to the preferences, for each criterion i, $i = 1, \ldots, n$, we consider t_i representative points well distributed on the scale of the criterion. Let us denote by γ_{ir} the r-th point, $r = 1, \ldots, t_i$, on the scale of criterion i. For each criterion i, the DM is asked to supply $t_i\frac{t_i-1}{2}$ pairwise comparisons between points γ_{ir} and γ_{is}, $r, s = 1, \ldots, t_i$. Using AHP, for each representative point γ_{ir}, the normalized evaluations

$u(\gamma_{ir})$ are obtained. Consider now an alternative a having an evaluation $g_i(a)$ on criterion i. If $g_i(a)$ belongs to the interval of consecutive representative points $\left[\gamma_{is}, \gamma_{i(s+1)}\right]$, we can get the normalized evaluation $u(g_i(a))$ of $g_i(a)$ as follows:

$$u(g_i(a)) = u(\gamma_{is}) + \frac{u(\gamma_{i(s+1)}) - u(\gamma_{is})}{\gamma_{i(s+1)} - \gamma_{is}}(g_i(a) - \gamma_{is}). \tag{3.23}$$

It is to observe that the selected set of points may have an influence on the results of the rescaling. They must therefore be selected carefully and in the most representative way. We believe that the best way to select reference points is to select them in cooperation with the DM. Indeed, the fundamental characteristic that the reference points must possess is their meaningfulness for the DM. In addition to the greater reliability of her comparisons, there is another advantage in involving the DM in the selection of the reference points. Indeed, the more the reference points are meaningful for the DM, the more she will feel comfortable in reasoning about possible inconsistencies and correct them, if necessary.

With respect to the procedure we are proposing, one could ask if the reduction of comparisons does not reduce the possibility of discovering inconsistencies in the DM's evaluations, with the risk of obtaining a decision model not enough accurate and convincing. We do not believe that this is the case. Indeed, asking for comparing all the alternatives, especially in case of a great number of alternatives, the DM experiments a great cognitive burden which is not compensated by a greater reliability of the results. It is rather the contrary. The greater the number of the questions requested to the DM and the more difficult the contents of these questions (because very often are based on negligible differences), the less is the attention in answering and the more deteriorate the overall quality of the preference information collected by the DM. Let us observe that this consideration is confirmed by the growing literature in the domain of the heuristic decision making (see e.g. [65]) which is mainly based on the principle that ignoring part of the information can lead to more accurate and more effective decisions. Moreover, even supposing an "ideal" DM able to answer in a good way to all the many questions one can ask her, the traditional AHP is not adapted for large problems. Observe that some studies even recommend only 7 alternatives because of the limited capacity of our brain to compare more alternatives [149]. Therefore, reducing reasonably the number of pairwise comparisons will not reduce the quality of the obtained decision model because the traditional AHP will anyway have a reduced quality with a high number of alternatives and, probably, it will also increase the reliability of the final results.

Since AHP has received several criticisms, one could wonder how they apply to the methodology

we are proposing. We believe that the soundness of our application of AHP is not touched by these remarks. For example, we agree with Salo and Hämäläinen [152] that the 1-9 scale is not always justified. However, recent publications [96, 139] have proposed to calibrate the measurement scale to the mental representation of the DM. We have not done this extra step because it is not the main focus of the paper, but it can easily be added. Also the criticism of Perez, Jimeno and Mokotoff [134] regarding the use of indifferent criteria in AHP does not apply to our model. Indeed, in our work, we only use partially AHP. More precisely, we use only the pairwise comparison matrices for comparing performances but we do not use pairwise comparison matrices for finding the weights of the criteria since we use the NAROR technique for this part. Bana e Costa and Vansnick [10] observe that the Condition of Order Preservation, that is, loosely speaking, that priorities represent also intensity of preferences between alternatives, is not preserved in AHP. This is true if we consider direct evaluations only. As we consider useful to take into account direct and indirect evaluations to derive priorities [94, 176], also this criticism is not applicable to our model.
Finally, it is important to mention that the procedure we are proposing consisting of compare pairwise a small number of reference points only can also be applied to other methods that give a normalized evaluation on the basis of pairwise comparisons as, for example, MACBETH [9].

3.2.4 An application for the conjoint use of NAROR and AHP

Steve wants to buy an economy car and, therefore, he is analyzing currently available cars on the market. Thus, he decides to consider 24 models that are presented in Table 3.19 together with their evaluations with respect to criteria Price, Acceleration, Max Speed and Fuel Consumption. First of all, in order to use the Choquet integral preference model and the NAROR to decide which car to buy, the evaluations of each car with respect to above criteria have to be expressed on a common scale. This is possible using AHP but it requires 276 pairwise comparisons for each one of the considered criteria which lead to a total of 1104 pairwise comparisons. Steve thinks that all these pairwise comparisons are too many, and thus we propose him to apply AHP only to a set of representative evaluations on the scale of each criterion and to determine the normalized value of the evaluations of the 24 considered cars by using the procedure described in Section 3.2.3.
For this reason, we ask Steve to compare the representative values shown in Table 3.20. As a consequence, the pairwise comparisons asked to Steve are:

- 55 for the 11 reference levels of criterion Price,
- 28 for the 8 reference levels of criterion Acceleration,

Table 3.19: Set of considered cars and their evaluation

	Price [Euro]	Acceleration [seconds from 0 to 100 km/h]	Max Speed [km/h]	Consumption (l/km)
(a_1) **Audi A3 (3-doors)**	22,140	10.3	193	4.9
(a_2) **BMW 1 Series (3-doors)**	23,089	11.2	195	5.5
(a_3) **Hyunday ix20**	14,000	12.9	167	6
(a_4) **Ford C-Max**	19,500	12.6	174	5.1
(a_5) **Toyota Aygo**	10,350	13.7	157	4.4
(a_6) **Seat Ibiza (5-doors) Style**	13,000	13.9	163	5.4
(a_7) **VolksWagen Polo highline 1.4 (3-door)**	16,550	12.1	177	5.9
(a_8) **BMW Serie 1 (3-doors)**	23,069	11.2	195	5.5
(a_9) **Chevrolet Spark**	9,952	15.3	152	5
(a_{10}) **FIAT Punto (3-doors)**	13,711	11.2	182	4.2
(a_{11}) **Ford Fiesta (3-doors)**	12,750	14.9	165	4.6
(a_{12}) **Honda Civic**	18,900	13.4	187	5.4
(a_{13}) **Kia Rio**	11,650	13.1	172	5.1
(a_{14}) **Lancia Ypsilon**	14,568	11.9	176	4.2
(a_{15}) **Mazda2 3-door Sporty**	14,900	13.6	172	5
(a_{16}) **Mercedes A-Class**	23,630	9.2	202	5.5
(a_{17}) **Mini Cooper**	20,700	7.9	210	4.5
(a_{18}) **Mitsubishi Space Star**	11,490	13.6	172	4
(a_{19}) **Nissan Micra**	11,250	13.7	170	5
(a_{20}) **Opel Corsa**	11,330	18.2	155	5.1
(a_{21}) **Peugeot 208**	12,100	14	163	4.3
(a_{22}) **Renault Clio**	16,200	12.2	182	4.5
(a_{23}) **Skoda Citygo**	9,260	14.4	160	4.5
(a_{24}) **Suzuki Swift**	12,100	11.5	165	5

- 21 for the 7 reference levels of criterion Max Speed,
- 36 for the 9 reference levels of criterion Consumption,

which gives a total of 140 pairwise comparisons. The pairwise comparisons given by Steve are shown in Tables 3.21(a)-3.21(d) while the normalized evaluations of these reference points, obtained by AHP, are provided in Table 3.22. Considering the normalized evaluations of the reference points and interpolating them as described in the previous Section, we are able to obtain the normalized evaluations of all cars with respect to all criteria reported in Table 3.23. For example, to obtain the normalized evaluation of the Kia Rio with respect to price, first of all we have to observe that its price $(11,650$ euro), is in the interval of references prices whose extremes are $10,500$ euro and $12,000$ euro. Since the utilities of these reference prices obtained by AHP are respectively 0.2907 and 0.2489, applying (3.23) we get the normalized price of the Kia Rio as follows:

$$
\begin{aligned}
u(11,650) &= u(10,500) + \frac{u(12,000) - u(10,500)}{12,000 - 10,500}(11,650 - 10,500) \\
&= 0.2907 + \frac{0.2489 - 0.2907}{1,500}(1,150) = 0.2587.
\end{aligned}
$$

To apply the Choquet integral preference model, we decide to use the indirect preference information and, consequently, Steve provides some preference information about the interaction between criteria (the constraints translating the corresponding information are in brackets):

Table 3.20: Reference levels for considered criteria

Price [Euro]	Acceleration [seconds from 0 to 100 km/h]	Max Speed [Km/h]	Consumption (l/km)
9,000	7	150	4
10,500	9	160	4.25
12,000	10.5	170	4.5
13,500	12	180	4.75
15,000	13.5	190	5
17,000	15	200	5.25
19,000	16.5	210	5.5
21,000	19		5.75
23,000			6
25,000			

Table 3.21: Pairwise comparison matrix for the considered criteria

(a) Price (CI=0.1)

	9,000	10,500	12,000	13,500	15,000	17,000	19,000	21,000	23,000	25,000
9,000	1	9	9	9	9	9	9	9	9	9
10,500	1/9	1	2	3	3	4	6	7	9	9
12,000	1/9	1/2	1	2	3	5	6	8	9	9
13,500	1/9	1/3	1/2	1	2	3	3	4	6	9
15,000	1/9	1/3	1/3	1/2	1	2	4	4	6	9
17,000	1/9	1/4	1/5	1/3	1/2	1	2	3	4	6
19,000	1/9	1/6	1/6	1/3	1/4	1/2	1	2	3	5
21,000	1/9	1/7	1/8	1/4	1/4	1/3	1/2	1	2	4
23,000	1/9	1/9	1/9	1/6	1/6	1/4	1/3	1/2	1	2
25,000	1/9	1/9	1/9	1/9	1/9	1/6	1/5	1/4	1/2	1

(b) Acceleration (CI=0.05)

	7	9	10.5	12	13.5	15	16.5	19
7	1	3	4	6	7	7	8	9
9	1/3	1	2	4	5	6	7	9
10.5	1/4	1/2	1	2	4	5	6	8
12	1/6	1/4	1/2	1	2	3	4	6
13.5	1/7	1/5	1/4	1/2	1	2	3	5
15	1/7	1/6	1/5	1/3	1/2	1	2	4
16.5	1/8	1/7	1/6	1/4	1/3	1/2	1	2
19	1/9	1/9	1/8	1/6	1/5	1/4	1/2	1

(c) Speed (CI=0.1)

	150	160	170	180	190	200	210
150	1	1/5	1/6	1/7	1/8	1/8	1/8
160	5	1	1/2	1/6	1/7	1/7	1/7
170	6	2	1	1/2	1/6	1/7	1/7
180	7	6	2	1	1/5	1/5	1/5
190	8	7	6	5	1	1/2	1/3
200	8	7	7	5	2	1	1
210	8	7	7	5	3	1	1

(d) Fuel consumption (CI=0.08)

	4	4.25	4.5	4.75	5	5.25	5.5	5.75	6
4	1	3	4	5	6	7	8	9	9
4.25	1/3	1	2	4	5	6	7	8	9
4.5	1/4	1/2	1	2	3	5	6	8	9
4.75	1/5	1/4	1/2	1	2	4	5	6	7
5	1/6	1/5	1/3	1/2	1	2	4	5	7
5.25	1/7	1/6	1/5	1/4	1/2	1	2	4	6
5.5	1/8	1/7	1/6	1/5	1/4	1/2	1	2	7
5.75	1/9	1/8	1/8	1/6	1/5	1/4	1/2	1	3
6	1/9	1/9	1/9	1/7	1/7	1/6	1/7	1/3	1

Table 3.22: Reference levels for considered criteria and corresponding normalized values obtained by AHP

Price [Euro]	Norm.	Acceleration [seconds from 0 to 100 km/h]	Norm.	Max Speed [Km/h]	Norm.	Consumption (l/km)	Norm.
9,000	1	7	1	150	0	4	1
10,500	0.2907	9	0.5595	160	0.0616	4.25	0.6224
12,000	0.2489	10.5	0.3622	170	0.1096	4.5	0.4079
13,500	0.1454	12	0.1892	180	0.2432	4.75	0.2659
15,000	0.1167	13.5	0.1108	190	0.6438	5	0.1692
17,000	0.0661	15	0.0649	200	0.9110	5.25	0.0967
19,000	0.0396	16.5	0.0243	210	1	5.5	0.0574
21,000	0.0220	19	0			5.75	0.0211
23,000	0.0088					6	0
25,000	0						

- Price (P) and Acceleration (A) are positively interacting ($\varphi(P,A) \geq \varepsilon$),
- Price and Max Speed (M) are positively interacting ($\varphi(P,M) \geq \varepsilon$),
- Acceleration and Consumption (C) are positively interacting ($\varphi(A,C) \geq \varepsilon$),

Table 3.23: Set of considered cars with normalized evaluations on each criterion

	Price [Euro]	Acceleration [seconds from 0 to 100 km/h]	Max Speed [Km/h]	Consumption (l/km)
(a_1) **Audi A3 (3-doors)**	0.0145	0.3885	0.7240	0.2079
(a_2) **BMW 1 Series (3-doors)**	0.0084	0.2814	0.7774	0.0574
(a_3) **Hyunday ix20**	0.1358	0.1422	0.0952	0.0000
(a_4) **Ford C-Max**	0.0352	0.1578	0.1630	0.1402
(a_5) **Toyota Aygo**	0.3617	0.1047	0.0432	0.4937
(a_6) **Seat Ibiza (5-doors) Style**	0.1799	0.0986	0.0760	0.0731
(a_7) **VolksWagen Polo highline 1.4 (3-door)**	0.0775	0.1840	0.2031	0.0085
(a_8) **BMW Serie 1 (3-doors)**	0.0085	0.2814	0.7774	0.0574
(a_9) **Chevrolet Spark**	0.5499	0.0568	0.0123	0.1692
(a_{10}) **FIAT Punto (3-doors)**	0.1413	0.2814	0.3233	0.6979
(a_{11}) **Ford Fiesta (3-doors)**	0.1971	0.0679	0.0856	0.3511
(a_{12}) **Honda Civic**	0.0410	0.1160	0.5236	0.0731
(a_{13}) **Kia Rio**	0.2587	0.1317	0.1363	0.1402
(a_{14}) **Lancia Ypsilon**	0.1250	0.2007	0.1897	0.6979
(a_{15}) **Mazda2 3-door Sporty**	0.1186	0.1077	0.1363	0.1692
(a_{16}) **Mercedes A-Class**	0.0060	0.5332	0.9288	0.0574
(a_{17}) **Mini Cooper**	0.0247	0.8018	1.0000	0.4079
(a_{18}) **Mitsubishi Space Star**	0.2631	0.1077	0.1363	1.0000
(a_{19}) **Nissan Micra**	0.2698	0.1047	0.1096	0.1692
(a_{20}) **Opel Corsa**	0.2676	0.0078	0.0308	0.1402
(a_{21}) **Peugeot 208**	0.2420	0.0955	0.0760	0.5795
(a_{22}) **Renault Clio**	0.0863	0.1787	0.3233	0.4079
(a_{23}) **Skoda Citygo**	0.8771	0.0832	0.0616	0.4079
(a_{24}) **Suzuki Swift**	0.2420	0.2468	0.0856	0.1692

- Max Speed and Consumption are positively interacting ($\varphi(M,C) \geq \varepsilon$),
- Price and Consumption are negatively interacting ($\varphi(P,C) \leq -\varepsilon$),
- Acceleration and Max Speed are negatively interacting ($\varphi(A,M) \leq -\varepsilon$).

Steve provides also the following preference order on some cars that he knows and for which he is able to form his preferences:

$$a_3 \succ a_5 \succ a_6 \succ a_7 \succ a_1 \succ a_4 \succ a_2.$$

This preference information is translated by the following linear inequalities:

- $C_\mu(a_3) \geq C_\mu(a_5) + \varepsilon, \qquad C_\mu(a_5) \geq C_\mu(a_6) + \varepsilon, \qquad C_\mu(a_6) \geq C_\mu(a_7) + \varepsilon,$
- $C_\mu(a_7) \geq C_\mu(a_1) + \varepsilon, \qquad C_\mu(a_1) \geq C_\mu(a_4) + \varepsilon, \qquad C_\mu(a_4) \geq C_\mu(a_2) + \varepsilon,$

where ε is an auxiliary variable supposed being greater than zero.

To check if there exist at least one capacity compatible with the preferences Steve provided us, we have to solve the following linear programming problem:

$$
\varepsilon^* = \max \varepsilon, s.t.
$$

$$
\left.\begin{array}{l}
\left.\begin{array}{l}
C_\mu(a_3) \geq C_\mu(a_5) + \varepsilon, \quad C_\mu(a_5) \geq C_\mu(a_6) + \varepsilon, \\
C_\mu(a_6) \geq C_\mu(a_7) + \varepsilon, \quad C_\mu(a_7) \geq C_\mu(a_1) + \varepsilon, \\
C_\mu(a_1) \geq C_\mu(a_4) + \varepsilon, \quad C_\mu(a_4) \geq C_\mu(a_2) + \varepsilon, \\
m(\{P,A\}) \geq \varepsilon, \quad m(\{P,M\}) \geq \varepsilon, \\
m(\{A,C\}) \geq \varepsilon, \quad m(\{M,C\}) \geq \varepsilon, \\
m(\{P,C\}) \leq -\varepsilon, \quad m(\{A,M\}) \leq -\varepsilon,
\end{array}\right\} \\
m\left(\{\emptyset\}\right) = 0, \ \sum\limits_{i \in \{P,A,M,C\}} m\left(\{i\}\right) + \sum\limits_{\{i,j\} \ \subseteq \{P,A,M,C\}} m\left(\{i,j\}\right) = 1, \\
m\left(\{i\}\right) \geq 0, \ \forall i \in \{P,A,M,C\}, \\
m\left(\{i\}\right) + \sum\limits_{j \in T} m\left(\{i,j\}\right) \geq 0, \forall i \in \{P,A,M,C\} \text{ and } \forall\, T \subseteq \{P,A,M,C\} \setminus \{i\}, T \neq \emptyset,
\end{array}\right\}
$$

Because ε^* is greater than zero, this means that there exists at least one capacity compatible with the preference information provided by Steve. Table 3.24 presents the Möbius representation of the capacity corresponding to ε^*.

Table 3.24: Möbius representation of the capacity maximizing the value of ε

$\mathbf{m}(\{P\})$	$\mathbf{m}(\{A\})$	$\mathbf{m}(\{M\})$	$\mathbf{m}(\{C\})$	$\mathbf{m}(\{P,A\})$	$\mathbf{m}(\{P,M\})$	$\mathbf{m}(\{P,C\})$	$\mathbf{m}(\{A,M\})$	$\mathbf{m}(\{A,C\})$	$\mathbf{m}(\{M,C\})$
0.0380	0.0833	0.0133	0.0380	0.7374	0.0809	-0.0380	-0.0133	0.0133	0.0472

On the basis of the Möbius representation of the capacity maximizing ε, we can compute the Shapley index of each criterion. For example, the Shapley index of criterion price will be obtained as follows:

$$
\varphi(\{P\}) = m(\{P\}) + \frac{m(\{P,A\}) + m(\{P,M\}) + m(\{P,C\})}{2} = 0.0380 + \frac{0.7374 + 0.0809 - 0.0380}{2} = 0.4281.
$$

Looking at Tables 3.24 and 3.25, one can see that Acceleration is the most important criterion both considered singularly (because $m(\{M\}) > m(\{i\}), i \in \{P,A,C\}$), and also when taking into account its interactions with the other three criteria (because $\varphi(\{M\}) > \varphi(\{i\}), i \in \{P,A,C\}$). On the other hand, Maximum Speed is the less important criterion if it is considered singularly, while Consumption is the less important criterion considering also all its interactions with the other criteria.

Table 3.25: Importance of criteria measured by the Shapley index

	φ
$\{\mathbf{P}\}$	0.4281
$\{\mathbf{A}\}$	0.452
$\{\mathbf{M}\}$	0.0707
$\{\mathbf{C}\}$	0.0492

In order to take into account not only one but the whole set of capacities compatible with the preference information provided by Steve, we apply the NAROR as described in Section 3.2.3. In Table 3.26, we reported the necessary preference relation $\succsim^N$, while in Table 3.27 we reported the asymmetric part ($\succ^P$) of the possible preference relation $\succsim^P$, where $a \succ^P b$ iff $a \succsim^P b$ and $not(b \succsim^P a)$. We presented $\succ^P$ instead of $\succsim^P$, because $\succ^P \subseteq \succsim^N \subseteq \succsim^P$ (that is $\succ^P$ is much more synthetic than $\succsim^P$) and, overall, because $\succ^P$ is transitive, which is not the case for $\succsim^P$ [86]. In simple words, $\succ^P$ is much more intelligible than $\succsim^P$ for the DM.

Table 3.26: Necessary preference relation: for each line, the car on the left is necessarily preferred to the cars on the right. For example, a_1 is necessarily preferred to a_2, a_4 and a_8 while the viceversa is not true.

	Cars
(a_1) **Audi A3 (3-doors)**	a_2, a_4, a_8,
(a_2) **BMW 1 Series (3-doors)**	$\emptyset$
(a_3) **Hyunday ix20**	a_1, a_2, a_4, a_5, a_6, a_7, a_8, a_{12}, a_{20},
(a_4) **Ford C-Max**	a_2,
(a_5) **Toyota Aygo**	a_1, a_2, a_4, a_6, a_7, a_8, a_{12}, a_{20},
(a_6) **Seat Ibiza (5-doors) Style**	a_1, a_2, a_4, a_7, a_8, a_{12}, a_{20},
(a_7) **VolksWagen Polo highline 1.4 (3-door)**	a_1, a_2, a_4, a_8, a_{12}, a_{20},
(a_8) **BMW Serie 1 (3-doors)**	a_2
(a_9) **Chevrolet Spark**	$\emptyset$
(a_{10}) **FIAT Punto (3-doors)**	a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{11}, a_{12}, a_{14}, a_{15}, a_{16}, a_{19}, a_{20}, a_{21}, a_{22},
(a_{11}) **Ford Fiesta (3-doors)**	a_2, a_4, a_8, a_{20},
(a_{12}) **Honda Civic**	$\emptyset$
(a_{13}) **Kia Rio**	a_1, a_2, a_4, a_5, a_6, a_7, a_8, a_9, a_{11}, a_{12}, a_{15}, a_{16}, a_{19}, a_{20}, a_{21}
(a_{14}) **Lancia Ypsilon**	a_1, a_2, a_4, a_6, a_7, a_8, a_9, a_{11}, a_{12}, a_{13}, a_{15}, a_{16}, a_{20}, a_{21}
(a_{15}) **Mazda2 3-door Sporty**	a_1, a_2, a_4, a_6, a_7, a_8, a_{11}, a_{12}, a_{20}
(a_{16}) **Mercedes A-Class**	$\emptyset$
(a_{17}) **Mini Cooper**	a_1, a_2, a_4, a_8, a_{12}, a_{16}
(a_{18}) **Mitsubishi Space Star**	a_1, a_2, a_4, a_6, a_7, a_8, a_9, a_{11}, a_{12}, a_{15}, a_{19}, a_{20}, a_{21}
(a_{19}) **Nissan Micra**	a_1, a_2, a_4, a_6, a_7, a_8, a_9, a_{11}, a_{12}, a_{20}
(a_{20}) **Opel Corsa**	$\emptyset$
(a_{21}) **Peugeot 208**	a_2, a_4, a_8, a_{12}, a_{20},
(a_{22}) **Renault Clio**	a_1, a_2, a_4, a_7, a_8, a_{11}, a_{12} a_{20}
(a_{23}) **Skoda Citygo**	a_2, a_4, a_8, a_9, a_{12}, a_{20}
(a_{24}) **Suzuki Swift**	a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{12}, a_{16}, a_{20}

To summarize the results of ROR, we build also the most representative model, being that one that maximally discriminate between alternatives for which there is a necessary preference. By computing the Choquet integral preference model of each alternative considering the most representative model, we obtain the following final ranking (in brackets the corresponding value obtained by applying the

Table 3.27: Asymmetric part of the possible preference relation: for each line, the car on the left is possibly preferred to the cars on the right, which, in turn, are not possibly preferred to the car on the left. For example, a_1 is possibly preferred to a_2, a_4 and a_8, while no one between a_2, a_4 and a_8 is possibly preferred to a_1.

	Cars
(a_1) **Audi A3 (3-doors)**	$a_2, a_4, a_8,$
(a_2) **BMW 1 Series (3-doors)**	$\emptyset$
(a_3) **Hyunday ix20**	$a_1, a_2, a_4, a_5, a_6, a_7, a_8, a_{12}, a_{20},$
(a_4) **Ford C-Max**	$a_2,$
(a_5) **Toyota Aygo**	$a_1, a_2, a_4, a_6, a_7, a_8, a_{12}, a_{20},$
(a_6) **Seat Ibiza (5-doors) Style**	$a_1, a_2, a_4, a_7, a_8, a_{12}, a_{20},$
(a_7) **VolksWagen Polo highline 1.4 (3-door)**	$a_1, a_2, a_4, a_8, a_{12}, a_{20},$
(a_8) **BMW Serie 1 (3-doors)**	$a_2,$
(a_9) **Chevrolet Spark**	$\emptyset$
(a_{10}) **FIAT Punto (3-doors)**	$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{11}, a_{12}, a_{14}, a_{15}, a_{16}, a_{19}, a_{20}, a_{21}, a_{22}, a_{22},$
(a_{11}) **Ford Fiesta (3-doors)**	$a_2, a_4, a_8, a_{20},$
(a_{12}) **Honda Civic**	$\emptyset$
(a_{13}) **Kia Rio**	$a_1, a_2, a_4, a_5, a_6, a_7, a_8, a_9, a_{11}, a_{12}, a_{15}, a_{16}, a_{19}, a_{20}, a_{21}$
(a_{14}) **Lancia Ypsilon**	$a_1, a_2, a_4, a_6, a_7, a_8, a_9, a_{11}, a_{12}, a_{15}, a_{16}, a_{20}, a_{21},$
(a_{15}) **Mazda2 3-door Sporty**	$a_1, a_2, a_4, a_6, a_7, a_8, a_{11}, a_{12}, a_{20},$
(a_{16}) **Mercedes A-Class**	$\emptyset$
(a_{17}) **Mini Cooper**	$a_1, a_2, a_4, a_8, a_{12}, a_{16},$
(a_{18}) **Mitsubishi Space Star**	$a_1, a_2, a_4, a_6, a_7, a_8, a_9, a_{11}, a_{12}, a_{15}, a_{19}, a_{20}, a_{21},$
(a_{19}) **Nissan Micra**	$a_1, a_2, a_4, a_6, a_7, a_8, a_9, a_{11}, a_{12}, a_{20},$
(a_{20}) **Opel Corsa**	$\emptyset$
(a_{21}) **Peugeot 208**	$a_2, a_4, a_8, a_{12}, a_{20},$
(a_{22}) **Renault Clio**	$a_1, a_2, a_4, a_7, a_8, a_{12}, a_{20},$
(a_{23}) **Skoda Citygo**	$a_2, a_4, a_8, a_9, a_{12}, a_{20},$
(a_{24}) **Suzuki Swift**	$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{12}, a_{16}, a_{20}$

most representative model):

$$a_{24}(0.2415) \succ a_{10}(0.1520) \succ a_3(0.135) \succ a_{14}(0.132) \succ a_{13}(0.1319) \succ a_{18}(0.1156) \succ a_{15}(0.1092) \succ$$

$$\succ a_5(0.1076) \succ a_{19}(0.1054) \succ a_{21}(0.0992) \succ a_6(0.0984) \succ a_{22}(0.0971) \succ a_{23}(0.0858) \succ a_7(0.0813) \succ$$

$$\succ a_{11}(0.0707) \succ a_{17}(0.0614) \succ a_{12}(0.0579) \succ a_9(0.0576) \succ a_1(0.0406) \succ a_4(0.0405) \succ a_{16}(0.0384) \succ$$

$$\succ a_8(0.0355) \succ a_2(0.0354) \succ a_{20}(0.0096)$$

From this representative ranking, we can state that a_{24} is the best car, while the worst is surely a_{20}.

3.2.5 Conclusions

We considered multiple criteria decision aiding in case of interaction between criteria using the Choquet integral preference model. The application of the Choquet integral preference model requires that evaluations with respect to considered criteria are expressed on a common scale. We used AHP to build this common scale taking into account preference information given by the DM. To reduce considerably the number of pairwise comparisons usually required to the DM when applying AHP, we proposed to use AHP on a set of few reference points in the scale of each criterion and to interpolate the other values. Then, we adopted the recently introduced NAROR taking into account all the capacities compatible with the preference information provided by the DM. We illustrated the conjoint use of NAROR and AHP with an application to the decision problem of choosing a car to buy. We believe that the procedure we are proposing conjugates harmoniously the advantages of AHP in building a measurement scale and the advantages of the Choquet integral in handling interaction between criteria. In this context, the adoption of NAROR seems very beneficial because it permits to avoid focusing on only one capacity, which can be misleading for the reliability of the final decision. We believe also that the procedure of applying directly AHP on only a small set of reference points and using the linear interpolation to get the other values deserves to be considered generally in future applications, regardless from the use of the Choquet integral. Indeed, the high number of pairwise comparisons requested to the DM to apply AHP is a problem that can prevent its application in case of too many alternatives. Moreover, the request of too many pairwise comparisons can limit the reliability of the results supplied by AHP. In this perspective, our proposal can result very useful for application of AHP in all the many real life decision problems presenting a high number of alternatives. We conclude with an interesting extension of our method that we plan to develop in a future paper. It regards consideration of the probability that a given alternative has a certain rank or the probability that an alternative is preferred to another taking into consideration the whole set of compatible capacities according to the methodology proposed in [4].

3.3 Using Choquet Integral as Preference Model in Interactive Evolutionary Multiobjective Optimization

3.3.1 Introduction

Multiobjective optimization involves several conflicting objectives that compete for the best solution in a constrained multidimensional space of decision variables. In general, there is no single optimal solution (as in single-objective optimization), but a set of alternatives for which it is not possible to improve one objective without deteriorating another one, called Pareto-optimal solutions. Despite the existence of multiple Pareto-optimal solutions, in practice, usually only one of these solutions is to be chosen. Thus, in multiobjective optimization, there are two equally important tasks: an optimization task for finding Pareto-optimal solutions by a search procedure, and a decision aiding task for recommending a single most preferred solution. The "most preferred" refers to the value system of a particular user, also called decision maker (DM). Thus, decision aiding necessitates some preference elicitation from the user.

As to procedures searching for Pareto-optimal solutions, in the last two decades we have been able to observe a growing popularity of algorithms adopting the principles of natural evolution. A distinguishing feature of these evolutionary algorithms is that they work with a population of solutions. This is of particular advantage in the case of multiobjective optimization, as they can search for several Pareto-optimal solutions simultaneously in one run, providing the user with a set of feasible solutions to choose from. In the early stage of development of evolutionary algorithms for multiobjective optimization, the efforts were focused on efficient generation of the whole set of Pareto-optimal solutions (or of a good approximation thereof), leaving the decision aiding task to the post-optimization stage. Later, researchers started interlacing the optimization and preference handling in interactive procedures, which allows to converge more quickly to the most preferred region of the Pareto-optimal front [21].

Most interactive procedures for multiobjective evolutionary optimization assume a particular mathematical model of user's preferences. This model drives the search procedure towards the most preferred Pareto-optimal solutions. The model building involves preference information supplied by the user. In case of simple preference models, one may expect that the user can provide directly the values of model parameters. However, simple models, like the weighted sum of objectives, fail to represent more subtle user's preferences (see Section 3.3.4). For this reason, there is a tendency to use more complex preference models built from indirect preference information which is much easier to elicit by the user than the direct preference information. In many previous studies on interactive multiobjective optimization, the indirect preference information had the form of pairwise comparisons of some solutions from a current population [23, 24, 91, 135]. In particular, the NEMO-I[3] method [23] is defining a set of value functions compatible with the preferences elicited from the DM, expressed in terms of pairwise comparisons of solutions. These compatible value functions are used in Robust Ordinal Regression (ROR) [35, 86] to build a *necessary preference relation* ($\succsim^N$) on the current population of solutions. More precisely, $a \succsim^N b$ if a is at least as good as b for all compatible value functions. NEMO-I is adopting the scheme of NSGA-II [39], however substituting the dominance relation by the necessary preference relation in the ranking. While NEMO-I has shown a satisfactory convergence to the best compromise solution, the calculation of the necessary preference relation requires a considerable computational effort. Therefore, in this paper, we are using an alternative method called NEMO-II, which overcomes the problem of prohibitive computational effort. NEMO-II accepts any type of value function. We are considering four types of value functions within NEMO-II: linear, additive piecewise-linear, general additive, and (for the first time in combination with evolutionary multiobjective algorithms) the Choquet integral [31, 68].

The paper is organized as follows. In Section 2, we review interactive evolutionary multiobjective algorithms (MOEAs). Then, in Section 3, we present the general scheme of NEMO-II. Further, in Section 4, we describe the four above mentioned value functions. The new procedure, called NEMO-II-Ch, is introduced in Section 5. A computational experiment with the proposed procedure and its main competitors on a set of benchmark problems is presented in Section 6. Section 7 summarizes our conclusions and suggests avenues for future research.

3.3.2 Interactive Evolutionary Multiobjective Optimization

Evolutionary multiobjective optimization (EMO) has become very popular because of its ability to generate a set of non-dominated solutions in one run, from which the DM can choose a favorite solution without eliciting any preference information a priori. Nonetheless, in recent years, there has been a growing interest in EMO algorithms that are able to take into account user's preference information in the search process. This is motivated by the following expected advantages.

1. Instead of a diverse set of solutions (many of them clearly irrelevant to the DM) a search based on the DM's partial preferences will provide a more suitable sample of all Pareto-optimal solutions. It could either be a smaller set of only the most relevant solutions, or a more

[3]NEMO: Necessary preference enhanced Evolutionary Multiobjective Optimizer.

fine-grained resolution of the relevant parts of the Pareto frontier.

2. By focusing the search onto the relevant part of the search space, one may expect the optimization algorithm to find these solutions more quickly.

3. As the number of objectives increases, it becomes more and more difficult to identify and represent the complete Pareto-optimal frontier. This is partly because of the increasing number of Pareto-optimal solutions, but also because with an increasing number of objectives almost all solutions in the population become non-dominated, rendering dominance as selection criterion useless. User's preference information allows re-introducing the necessary selection pressure.

The literature contains today quite a few techniques that allow the incorporation of full or partial preference information into MOEAs, and previous surveys on this topic include [19, 20, 32, 33, 137].

Many of the techniques integrate partial user's preferences a priori, e.g., by allowing the DM to specify a reference point [58, 99, 151], maximal and minimal trade-offs [25], or desirability functions [173]. In the following, we focus on the literature that is most related to our paper, namely *interactive* approaches that learn user's preferences over the course of the optimization based on *the DM's (partial) ranking of small sets of solutions*. They allow to accumulate preference information and thus refine the internal preference model over time, and because they engage the DM in the optimization process, they initiate a learning process on the DM's side as well. These approaches are surveyed in the following, divided into methods that attempt to learn a representative user's value function, and those that focus on the set of value functions compatible with the elicited preference information. Some interactive approaches based on other paradigms include [40, 107, 151, 169].

We will use the following notation: A is the set of solutions in a considered population; $A^R \subseteq A$ is the set of reference solutions in population A; $f_1, \ldots, f_n$ are n objective functions such that each solution $a \in A$ is associated with the vector of evaluations $(f_1(a), \ldots, f_n(a))$. For simplicity, we sometimes only write a_j instead of $f_j(a)$. Unless specified otherwise, we suppose, without loss of generality, that the objective functions have to be maximized.

Approaches to learn a value function representing user's preferences

The approaches in this subsection use the elicited preference information to derive a single value function to approximate user's preferences. Value functions can have different complexity, ranging from simple linear functions to the highly non-linear functions considered in the non-parametric approaches, such as artificial neural networks or support vector machines. Most approaches simply use the derived value function for ranking individuals, sometimes as secondary criterion after non-dominance, but other uses can also be found.

Phelps and Köksalan [135] proposed an interactive evolutionary algorithm that periodically asks the DM to compare pairs of solutions. Assuming **linear** value functions (actually, the objectives are modified before the optimization to the squared distance from a reference value, which effectively results in ellipsoidal iso-utility curves), the method determines the most discriminant weight vector compatible with the preference information. Most discriminant here means the weight vector that maximizes the minimum value difference over all pairs of solutions compared by the DM.

Denote by $\succ_p$ the binary relation on the set A^R, representing the preference information provided by the user in terms of pairwise comparisons. Then, the following linear program (LP) identifies the most discriminant value function:

$$\left.\begin{aligned}&\max \varepsilon, \quad \text{subject to}\\ &\sum_{j=1}^{n} w_j f_j(a) - \sum_{j=1}^{n} w_j f_j(b) \geq \varepsilon, \quad \text{for all} \;\; a \succ_p b\\ &\sum_{j=1}^{n} w_j = 1, \quad w_j \geq 0.\end{aligned}\right\} \tag{3.24}$$

The resulting weight vector is then used for ranking individuals in the evolutionary algorithm that works as a single objective evolutionary algorithm between user interactions. If the LP is overconstrained and no feasible solution is found, the oldest preference information is discarded until feasibility is restored.

Deb et al. [41] derive a **polynomial** value function model. The user is shown a set of (five in the paper) solutions and asked to (at least partially) rank them. Then, similar to the approach by Phelps and Köksalan [135], the most discriminant value function is determined. However, due to the polynomial value function model, fitting the model to the specified preferences is a non-linear optimization problem, and the authors propose to use sequential quadratic programming to solve it. The most discriminant value function is used in the MOEA's ranking of individuals. Basically, the objective space is separated into two areas: all individuals with an estimated value (according to the approximated value function) better than the solution ranked *second* by the DM are assumed to dominate all the solutions with an estimated value worse than the solution ranked second. The authors additionally use the approximated value function to perform a local single-objective optimization starting with the solution ranked best by the DM.

Todd and Sen [170] use **artificial neural networks** to represent the DM's value function. Periodically, they present the DM with a set of solutions and ask for a score. The set of solutions is chosen such that they represent a broad variety regarding the approximated value function, and the estimated best and worst individual of the population are always included. Information from several interactions is accumulated after normalizing preference scores.

Another model that allows representation of complex value functions are **support vector machines** (SVM). Battiti and Passerini [11] use SVMs in the setting of an interactive MOEA. Periodically, the DM is presented with a set of solutions and asked to (at least partially) rank them. This information is then used to train the SVM, with cross-validation employed to select an appropriate kernel. The derived approximate value function is then used to sort individuals in the same non-dominance rank based on their value according to the learned value function. The paper examines the influence of the number of solutions shown to the DM (assuming full ranking) and the number of interactions with the DM. The results suggest that a relatively large number of solutions need to be ranked for the SVM to learn a useful value function (around 10-20), but only two interactions with the DM seem sufficient to come very close to results that would have been obtained had the DM's "*true value function*" been known from the beginning. The authors recommend to not start interaction until the MOEA has found a reasonable coverage of the entire Pareto frontier, which somewhat defeats the purpose of narrowing down the search early on. In [28], the approach's robustness to incorrect (noisy) DM preferences is examined and it is shown that the algorithm can cope well with noise.

Branke et al. [24] have recently compared various ways to define a representative value function and found that the function that maximizes the sum of values of individuals in the population actually performed slightly better than the most discriminant value function, and much better than a value function that minimizes slope changes. They showed that their approach, called NEMO-0, which is able to learn arbitrary monotonic additive value functions, can perform well in cases where a linear value function model is not sufficient to represent the user's preferences.

Approaches to learn a set of value functions representing user's preferences

Rather than deriving a single value function, Jaszkiewicz [100] notes that there may be several value functions compatible with the specified user's preferences and samples the preference function used in each generation from the set of preference functions (in this case **linear** weightings are assumed). The proposed approach uses the value function also for local search. In the interactive version, preference information from pairwise comparisons of solutions is used to reduce the set of possible weight vectors.

Greenwood et al. [91] suggested the imprecise value function approach which considers all compatible **linear** value functions *simultaneously*. The procedure asks the user to rank a few solutions, and from this derives constraints for the weightings of the objectives consistent with the given ordering. Then, these are used to check whether there is a feasible linear weighting such that solution a would be preferred to solution b.

Then, to compare any two solutions a and b from set A, one has to consider all value functions compatible with the user's preferences: a is considered at least as good as b if for all compatible value functions a gets a value not smaller than b. To make this conclusion, the following linear program (LP) has to be solved:

$$\left.\begin{array}{l} \varepsilon^* = \max \varepsilon, \quad \text{subject to} \\ \sum_{j=1}^{n} w_j f_j(b) - \sum_{j=1}^{n} w_j f_j(a) \geq \varepsilon \\ \sum_{j=1}^{n} w_j f_j(c) - \sum_{j=1}^{n} w_j f_j(d) \geq \varepsilon, \quad \text{for all } \; c \succ_p d \\ \sum_{j=1}^{n} w_j = 1, \quad w_j \geq 0. \end{array}\right\} E^N(a,b) \tag{3.25}$$

If the set of constraints $E^N(a,b)$ is infeasible or $\varepsilon^* \leq 0$, it can be concluded that there is no compatible value function such that b would be strictly preferred to a, and therefore a is at least as good as b for all compatible value functions. If $\varepsilon^* > 0$, then we know that b is possibly preferred over a, and we would proceed by checking whether b is always at least as good as a by solving $E^N(b,a)$. Overall, the method requires to solve one or two LPs for each pair of solutions in the population.

In [112], the value function model is only implicit. Under the assumption of **quasi-concave** value functions, specified preferences between solutions can be generalized to preference cones. This idea is used by Fowler et al. [59] to partially rank the non-dominated solutions in an MOEA. The DM is asked to consider a set of six solutions and specify the best and worst. From this information, six preference cones are derived (five 2-point cones involving the best and any of the other solutions, and one 6-point preference cone specifying that five solutions are better than the worst). All generated cones are kept throughout the optimization run, even if the solutions defining the cone are deleted from the population. The solutions shown to the DM are selected from the set of non-dominated solutions that cannot already be ranked with the existing cones.

Branke et al. [22, 23] proposed a method called NEMO-I. It is similar to the imprecise value function approach by Greenwood et al. [91], but rather than being restricted to linear value functions, it allows for **piecewise-linear** [23] or **general monotonic additive** [22, 23] value functions. NEMO-I replaces the use of the dominance relation in the non-dominance sorting step of NSGA-II by the necessary preference relation. Additionally, it computes a representative value function used for scaling in the crowding distance calculation.

The procedure used in NEMO-I is computationally very expensive because it requires solving at least one LP for each pair of solutions $(a, b) \in A \times A$. This means that, for a population composed of s solutions, one has to solve up to $s(s-1)$ LP problems in each iteration where we get new preference information. For this reason, in this paper, we are using a new variant, called NEMO-II, which requires significantly less computational effort. It has first been proposed in [24] as part of a general framework but it is implemented here for the first time.

3.3.3 General scheme of NEMO-II

In this section, we shall introduce the NEMO-II method presented as Algorithm 1. The procedure starts, as a classical MOEA, by randomly generating a population of solutions. Then, after ordering the population into fronts by using the dominance relation, two solutions in the first non-dominated front are chosen randomly to be compared by the user. The preference of the user on this pair of solutions is converted into a linear inequality involving the unknown value function. After introducing these constraints to the set of constraints defining the set of value functions compatible with the user's preferences, one has to check if the augmented set of constraints is feasible, which means that there exists at least one value function compatible with the preferences of the user. If this is not the case, a sufficient number of constraints being the cause of the inconsistency should be removed. For this reason, the procedure starts with removing the constraints representing the oldest pairwise comparisons. After the set of constraints becomes feasible, the method tries to reintroduce the removed constraints that were not the cause of the infeasibility in a reverse order (therefore starting from the newest pairwise comparison) as long as the feasibility is maintained.

The ordering of the population is done as in NSGA-II, putting the solutions into different fronts but, differently from NSGA-II, NEMO-II does not use the dominance relation. For each solution a in the current population A, NEMO-II checks whether there exists at least one compatible value function for which a is the most preferred solution in the current population, by solving the following LP problem

$$
\left.
\begin{array}{ll}
\varepsilon_a = \max \varepsilon, \;\; \text{subject to} & \\
U(a) - U(b) \geq \varepsilon \;\; \text{for all } b \in A \setminus \{a\}\,, & [C1] \\
U(c) - U(d) \geq \varepsilon \;\; \text{for all } \;c \succ_p d, & [C2] \\
\text{monotonicity and normalization constraints}, & [C3]
\end{array}
\right\} E_a
$$

where constraints [C1] are used to impose that a is the most preferred among the considered solutions; constraints [C2] translate the preferences of the user, while constraints [C3] are monotonicity and normalization constraints depending on the type of the adopted value function that shall be described in the next section.

If E_a is feasible and $\varepsilon_a > 0$, then there exists at least one value function for which the solution a is the most preferred solution and, therefore, it is included in the first front. After removing all solutions going in the first front, the same procedure is applied to build the other fronts until each solution is assigned to a front. Within the same front, the solutions are ordered by using the crowding distance on the objective space[4].

So, although NEMO-I and NEMO-II both take into account all value functions compatible with the user's preference information, NEMO-I makes pairwise comparisons between solutions, while NEMO-II compares each solution to all other solutions in the current population. The NEMO-II method has two advantages. First it substantially reduces the computational effort required. While in NEMO-I the construction of a front in a population of s individuals requires the solution of up to $s(s-1)$ LPs, in NEMO-II, the same can be done by solving only s LP problems, one for each solution in the population. Second, NEMO-II is slightly more precise in the sense that it only puts solutions in the best rank for which there exists a value function that makes them most preferred compared to all other solutions in the population, whereas NEMO-I may include some solutions for which no compatible value function exists that would prefer them over all other solutions, as long as no other solution is necessarily preferred in pairwise comparisons. Each front in NEMO-II is a subset of the front in NEMO-I that, in turn, is a subset of the non-dominated front.

A small example may illustrate the difference. Consider the case of three solutions a, b, c, evaluated with respect to the value functions U^1 and U^2 as follows: $U^1(a) = 1$, $U^1(b) = 0.5$ and $U^1(c) = 0$; $U^2(a) = 0$, $U^2(b) = 0.5$ and $U^2(c) = 1$. If U_1 and U_2 were the only value functions compatible with the user's preferences, NEMO-II would not put solution b into the first rank, as it would not be the most preferred under either value function. According to NEMO-I however, neither a nor c are

[4]The crowding distance of a solution a is the sum of distances between $a's$ left and right neighbor in each dimension, and infinity if a is an extreme solution [39].

necessarily preferred over b (because $U^2(b) > U^2(a)$ and $U^1(b) > U^1(c)$) and thus NEMO-I would put solution b into the first rank.

For these reasons, we base our new algorithm on the NEMO-II paradigm. The procedure is repeated until the assumed number of iterations has been reached.

Algorithm 1 Basic NEMO-II method

```
Generate initial population of solutions.
Elicit user's preferences by asking DM to compare two randomly selected non-dominated solutions.

Rank individuals into fronts by iteratively identifying all solutions that are most preferred for at
least one compatible value function. Rank within each front using crowding distance.
repeat
  Select individuals for mating.
  Generate offspring using crossover and mutation and add them to the population.
  if Time to ask the DM then
    Elicit user's preferences by asking DM to compare two randomly selected non-dominated
    solutions.
    if There is no value function remaining compatible with the user's preferences then
      Remove information on pairwise comparisons, starting from the oldest one, until feasibility
      is restored and reintroduce them in the reverse order as long as feasibility is maintained.
    end if
  end if
  Rank individuals into fronts by iteratively identifying all solutions that are most preferred for
  at least one compatible value function. Rank within each front using crowding distance.
  Reduce population size back to initial size by removing worst individuals.
until Stopping criterion met.
```

3.3.4 Preference Modeling Using Value Functions

Additive Preference Models

Among many preference models considered in the literature, the most popular is an *additive value function* defined on A, such that

$$U(a) = \sum_{j=1}^{n} u_j(f_j(a)) = \sum_{j=1}^{n} u_j(a_j), \tag{3.26}$$

where u_j are non-decreasing marginal value functions, $u_j : G_j \to \mathbb{R}$, $j \in G$, G_j is a value set of objective $f_j, j = 1, \ldots, n$, and G is the set of all indices of the objectives.

This model assumes one of the two forms of the marginal value functions $u_j(a_j)$:

(i) piecewise-linear,

(ii) general, non-decreasing.

In case (i), the ranges $[\alpha_j, \beta_j]$ are divided into $\gamma_j \geq 1$ sub-intervals $[a_j^0, a_j^1]$, $[a_j^1, a_j^2]$, ..., $[a_j^{\gamma_j-1}, a_j^{\gamma_j}]$, where $a_j^k = \alpha_j + \frac{k}{\gamma_j}(\beta_j - \alpha_j)$, $k = 0, \ldots, \gamma_j$, and $j \in G$, while α_j and β_j are the worst and the best performances on objective f_j, respectively. The marginal value of solution $a \in A$ with respect to objective f_j is obtained by linear interpolation,

$$u_j(a_j) = u_j(a_j^k) + \frac{a_j - a_j^k}{a_j^{k+1} - a_j^k}(u_j(a_j^{k+1}) - u_j(a_j^k)), \tag{3.27}$$

if $a_j \in [a_j^k, a_j^{k+1}]$, where $k \in \{0, ..., \gamma_j - 1\}$.

The piecewise-linear additive model is completely defined by the marginal values at the breakpoints, i.e., $u_j(a_j^0) = u_j(\alpha_j)$, $u_j(a_j^1)$, $u_j(a_j^2)$, $\cdots$, $u_j(a_j^{\gamma_j}) = u_j(\beta_j)$. Considering this type of preference function, one has to consider the following monotonicity and normalization constraints:

- for all $j \in G$ and for all $k = 0, \ldots, \gamma_j - 1$, $u_j(a_j^k) \leq u_j(a_j^{k+1})$;
- for all $j \in G$, $u_j(\alpha_j) = 0$, and $\sum_{j \in G} u_j(\beta_j) = 1$.

In case (ii), the characteristic points of marginal value functions $u_j, j \in G$, are fixed in evaluation points of considered solutions. Let τ_j be the permutation on the set of indices of solutions from A that reorders them according to non-decreasing evaluation on objective j, i.e.,

$$a_{\tau_j(1)} \leq a_{\tau_j(2)} \leq \ldots \leq a_{\tau_j(m-1)} \leq a_{\tau_j(m)}, \; j \in G, m = |A|.$$

The general non-decreasing additive model is completely defined by the marginal values at the characteristic points, i.e., $u_j(\alpha_j) = u_j(a_{\tau_j(1)})$, $u_j(a_{\tau_j(2)})$, $\ldots$, $u_j(a_{\tau_j(m)}) = u_j(\beta_j)$. Note that in this case, no linear interpolation is required to express the marginal value of any reference solution.

Considering this type of value function, the monotonicity constraints have the form $u_j(a_{\tau_j(k)}) \leq u_j(a_{\tau_j(k+1)})$, for all $k = 1, \ldots, m - 1$, while normalization constraints are the same as in case (i).

The Choquet Integral Preference Model

The simplest additive value function model is the weighted sum, obtained by assigning a non-negative weight w_j to each objective $f_j, j \in G$, and giving to each $a \in A$ the value

$$U(a) = \sum_{j=1}^{n} w_j f_j(a) = w_1 f_1(a) + \ldots + w_n f_n(a). \quad (3.28)$$

The weighted sum has some limitations in representing user's preferences, which are shown in the following example. Let us underline that in this example we shall speak of evaluation criteria, being in Multiple Criteria Decision Aiding (MCDA) the equivalent of the objective functions in Multiobjective Optimization.

Example. The manager of an international company wants to rank three candidates (*Smith*, *Johnson* and *Brown*), taking into account their performances on criteria experience (Ex) and age (Ag), given on a $[0, 10]$ scale (see Table 3.28).

Table 3.28: Experience (Ex) and age (Ag) of three candidates

	Ex	Ag
$Smith(S)$	6	10
$Johnson(J)$	8	8
$Brown(B)$	10	6

Since candidates having good experience are not necessarily young, and vice versa, if there is a good performance on one of the two criteria, one does not expect a good performance also on the other criterion. Consequently, a candidate being good both on experience and age is well appreciated. Therefore, in the manager's mind there is a positive interaction (synergy) between the performance on experience and the performance on age. In other words, the two criteria are not preferentially independent [109]. For this reason, the manager prefers *Johnson* to *Smith* and *Brown*.

If one would like to represent the preferences expressed by the manager using the weighted sum model, the following inequalities should be satisfied:

$$\begin{aligned} w_{Ex} \cdot 6 + w_{Ag} \cdot 10 &< w_{Ex} \cdot 8 + w_{Ag} \cdot 8, \\ w_{Ex} \cdot 10 + w_{Ag} \cdot 6 &< w_{Ex} \cdot 8 + w_{Ag} \cdot 8, \end{aligned}$$

where w_{Ex} and w_{Ag} are the weights of experience and age, respectively. It can be easily verified that the above inequalities are contradictory since:

$$w_{Ex} \cdot (-2) + w_{Ag} \cdot 2 < 0 < w_{Ex} \cdot (-2) + w_{Ag} \cdot 2.$$

Thus we have to conclude that, due to the positive interaction between the performances on experience and age, the weighted sum is not able to represent the manager's preferences. □

In order to represent preferences in case of interaction between criteria, non-additive integrals are often used [73]. The best-known non-additive integral in the literature is the Choquet integral [31]. The Choquet integral is based on the concept of capacity (fuzzy measure) that assigns a weight to each subset of criteria rather than to each single criterion. More precisely, denoting by 2^G the power set of G (i.e., the set of all subsets of G), the function $\mu : 2^G \rightarrow [0,1]$ is called capacity on 2^G if the following properties are satisfied:

1a) $\mu(\emptyset) = 0$ and $\mu(G) = 1$ (boundary conditions),

2a) $\forall\, T \subseteq S \subseteq G,\ \ \mu(T) \leq \mu(S)$ (monotonicity condition).

Intuitively, for all $T \subseteq G$, $\mu(T)$ can be interpreted as a comprehensive importance of the criteria from T considered as a whole.

Example (continuation). To represent the importance and the interaction of the performances on experience and age, one can set $\mu(\{Ex\}) = 0.4$, $\mu(\{Ag\}) = 0.3$ and $\mu(\{Ex, Ag\}) = 1$. The difference $\mu(\{Ex, Ag\}) - \mu(\{Ex\}) - \mu(\{Ag\}) = 0.3$ represents the positive interaction between experience and age because it measures how much greater is the importance of experience and age considered as a whole ($\mu(\{Ex, Ag\})$) comparing to the sum of their importances when they are considered separately ($\mu(\{Ex\})+\mu(\{Ag\})$). $\square$

The Choquet integral involving capacity μ assigns to each alternative $a \in A$ the following value:

$$C_\mu(a) = \sum_{j=1}^{n} \left[f_{(j)}(a) - f_{(j-1)}\,(a)\right] \mu\left(N_j\right), \tag{3.29}$$

where $_{(\cdot)}$ stands for a permutation of the indices of criteria, such that

$$f_{(0)}(a) \leq f_{(1)}\,(a) \leq f_{(2)}\,(a) \leq ... \leq f_{(n)}\,(a)\,, \tag{3.30}$$

$N_j = \{(j), \ldots, (n)\}$ and $f_{(0)}(a) = 0$.

Observe that (3.30) requires that the values taken by objective functions $f_j, j = 1, \ldots, n$, have to be non negative. If this is not the case, one can recode the values $f_j(a), a \in A$, with a translation $f_j^*(a) = f_j(a) + c$ with $c \geq -\min_{j \in G,\, b \in A} f_j(b)$, so that we get $f_j^*(a) \geq 0$ for all $j = 1, \ldots, n$ and all $a \in A$.

A meaningful reformulation of the capacity μ and of the Choquet integral can be obtained by means of the Möbius representation of the capacity μ (see [140]) which is a function $m : 2^G \to \mathbb{R}$ [153] defined as follows:

$$\mu(S) = \sum_{T \subseteq S} m(T).$$

Note that if S is a singleton, i.e., $S = \{j\}$ with $j = 1, \dots, n$, then $\mu(\{j\}) = m(\{j\})$. Moreover, if S is a pair of criteria, i.e., $S = \{i, j\}$, then $\mu(\{i, j\}) = m(\{i\}) + m(\{j\}) + m(\{i, j\})$.
The Möbius representation $m(S)$ of capacity $\mu(S)$ can be obtained as follows:

$$m(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \mu(T).$$

In terms of the Möbius representation, properties **1a)** and **2a)** are, respectively, restated as (see [30]):

1b) $m(\emptyset) = 0, \ \sum_{T \subseteq G} m(T) = 1,$

2b) $\forall\, j \in G$ and $\forall S \subseteq G \setminus \{j\}, \ \sum_{T \subseteq S} m(T \cup \{j\}) \geq 0,$

while the Choquet integral may be reformulated as follows:

$$C_\mu(a) = \sum_{T \subseteq G} m(T) \min_{j \in T} f_j(a). \tag{3.31}$$

Geometrically, in the case of two objectives ($G = \{f_1, f_2\}$), the iso-value curve of the Choquet integral can be decomposed into two linear functions, one above the line $f_1 = f_2$ and one below the line $f_1 = f_2$. Using (3.31) we can write

$$C_\mu(a) = \begin{cases} (m(\{1\}) + m(\{1,2\}))f_1(a) + m(\{2\})f_2(a), & \text{if } f_1(a) \leq f_2(a) \\ m(\{1\})f_1(a) + (m(\{2\}) + m(\{1,2\}))f_2(a), & \text{if } f_1(a) \geq f_2(a) \end{cases}$$

Example (continuation). The value assigned to *Smith* (S) by the Choquet integral with capacity μ is the following:

$$C_\mu(S) = f_{Ex}(S) \cdot \mu(\{Ex, Ag\}) + (f_{Ag}(S) - f_{Ex}(S)) \cdot \mu(\{Ag\}) = 7.2. \tag{3.32}$$

This value can be explained as follows. The performance $f_{Ex}(S) = 6$ is attained by the two criteria and thus it is multiplied by $\mu(\{Ex, Ag\})$ which is the weight assigned to experience and age considered as a whole. The performance $f_{Ag}(S) = 10$ is attained only on criterion age and therefore the difference $f_{Ag}(S) - f_{Ex}(S)$ is multiplied by $\mu(\{Ag\})$ which is the weight assigned to age considered alone. Analogously, we get $C_\mu(Johnson) = 8$ and $C_\mu(Brown) = 7.6$, so that we have $C_\mu(Johnson) > C_\mu(Smith)$ and $C_\mu(Johnson) > C_\mu(Brown)$, and thus we can conclude that the Choquet integral is able to represent the manager's preferences.

Observe, moreover, that the Möbius representation m of the capacity μ gives $m(\{Ex\}) = 0.4$, $m(\{Ag\}) = 0.3$ and $m(\{Ex, Ag\}) = 0.3$. Therefore, the Choquet integral referring to *Smith* can be reformulated as follows in terms of the Möbius representation m:

$$C_\mu(S) = f_{Ex}(S) \cdot m(\{Ex\}) + f_{Ag}(S) \cdot m(\{Ag\}) + \min(f_{Ex}(S), f_{Ag}(S)) \cdot m(\{Ex, Ag\}) = 7.2.$$

This value can be explained as follows. The performances on experience and on age are multiplied by $m(\{Ex\})$ and $m(\{Ag\})$, respectively, representing the relative weights of the two criteria. However, the value obtained by summation of the two weighted components has to be corrected by adding $\min(f_{Ex}(S), f_{Ag}(S)) \cdot m(\{Ex, Ag\})$ representing the positive interaction between the performance on experience and the performance on age. The Choquet integral of *Johnson* and *Brown* can be analogously reformulated in terms of the Möbius representation. □

In order to reduce the number of parameters to be elicited and to avoid an overprecise description of the interactions among criteria, Grabisch [69] introduced the concept of fuzzy k-additive capacity. A capacity is called k-additive if $m(T) = 0$ for $T \subseteq G$, such that $|T| > k$. In particular, in case of a 1-additive capacity, the Choquet integral is the standard weighted sum model.

In MCDA, it is easier and more straightforward to consider 2-additive capacities, since then the users have to express preference information on positive and negative interactions between two criteria only, neglecting possible interactions among three, four and, generally, r criteria, $r = 2, \ldots, n$. Moreover, by considering 2-additive capacities, the computational effort needed to determine the parameters is reduced since only $n + \binom{n}{2}$ coefficients have to be assessed; specifically, in terms of the Möbius representation, a value $m(\{i\})$ for every criterion i, and a value $m(\{i, j\})$ for every pair of criteria $\{i, j\}$. The value that a 2-additive capacity μ assigns to a set $S \subseteq G$ can be expressed in terms of the Möbius representation as follows:

$$\mu(S) = \sum_{i \in S} m(\{i\}) + \sum_{\{i,j\} \subseteq S} m(\{i,j\}), \ \forall S \subseteq G. \tag{3.33}$$

With regard to 2-additive capacities, properties **1b)** and **2b)** have, respectively, the following forms:

1c) $m(\emptyset) = 0, \ \sum_{i \in G} m(\{i\}) + \sum_{\{i,j\} \subseteq G} m(\{i,j\}) = 1,$

2c) $\begin{cases} m(\{i\}) \geq 0, \ \forall i \in G, \\ m(\{i\}) + \sum_{j \in T} m(\{i,j\}) \geq 0, \ \text{for all } i \in G, \ \text{and for all } T \subseteq G \setminus \{i\}, \ T \neq \emptyset. \end{cases}$

In this case, the Choquet integral of $a \in A$ is calculated as:

$$C_\mu(a) = \sum_{i \in G} m(\{i\}) f_i(a) + \sum_{\{i,j\} \subseteq G} m(\{i,j\}) \min(f_i(a), f_j(a)). \tag{3.34}$$

As one can observe, the use of the Choquet integral is based on several parameters (capacity $\mu(T)$ for each subset $T \subseteq G$ or a value $m(T)$ for each subset $T \subseteq G$ in case of the Möbius representation of capacity μ). To determine these parameters, a direct and an indirect technique known from the literature can be applied. In the direct technique, the user has to provide the parameters directly, while in the indirect technique the user has to provide some preference information from which parameters compatible with this information are retrieved by ordinal regression. The latter technique is much more realistic than the former because it requires less cognitive effort from the user. The indirect technique for the Choquet integral has been firstly proposed in [119]. When using the indirect technique, it is possible that more than one set of parameters is compatible with the preference information given by the user. For this reason, selection of only one of these compatible sets of parameters is somewhat arbitrary. To take into account all sets of parameters compatible with the user's preferences, Robust Ordinal Regression (ROR) [86] has been recently proposed. Taking into account all the sets of parameters compatible with the preferences of the user, ROR presents a recommendation in terms of *necessary* or *possible* preference relations which, for a pair of alternatives a and b, hold if a is at least as good as b for all or for at least one set of compatible parameters, respectively. ROR has been applied to the Choquet integral in [8] under the name of Non Additive Robust Ordinal Regression (NAROR).

Observe also that, besides determination of the capacity, the use of the Choquet integral involves another specific problem that is the construction of a common scale for the considered criteria permitting to compare the performances on different criteria and to compute in a meaningful way their difference. Indeed, looking at the definition of the Choquet integral in (3.29), we can observe that on one hand, permutation $_{(\cdot)}$ of the considered performances on different criteria is required, while on the other hand, the computation of the Choquet integral requires also that the differences between the performances on criteria $g_{(i)}$ and $g_{(i-1)}$, $i = 1, \ldots, n$, are meaningful. In the provided example this is obvious because, for instance, considering candidate *Smith*, the performance of 10 on age is clearly more valuable than the performance of 6 on experience, and their difference is 4. But, considering the case of a decision about cars, which is more valuable between a maximum speed of 200 km/h and a price of 35,000 euros? This means that in case of criteria with heterogeneous scales, the performances on all criteria have to be mapped to a common scale which permits to compare them and also to compute their difference. Very often a normalization of performances on each criterion is done considering an "unacceptable" and an "optimal value" for each criterion and considering a linear interpolation between these two extremes (see, e.g., [75]). A more sophisticated methodology permitting to construct a common scale and a capacity for the Choquet integral on the basis of preference information supplied by the DM has been proposed in [6] and further developed in [4]. In this paper, as we shall explain in detail in Section 3.3.5, we consider an intermediate approach consisting in first normalizing the performances on each criterion, and then rescaling them through multiplication by a set of weights that ensure comparability between performances on different criteria. We explain this procedure in the continuation of the previous example, in which the performances on considered criteria are already normalized but they need to be rescaled so that the preferences of the DM can be represented by the Choquet integral.

Example (continuation). Suppose that two new candidates have to be added to the three previously considered. Their performances with respect to experience and age are presented in table 3.29.

Table 3.29: Two new candidates evaluated on experience and age

	Ex	Ag
Baker	7	9
Miller	9	7

After reflecting a little, the manager arrived at the conclusion that he has the following preferences with respect to the five candidates:

$$Baker \succ Johnson \succ Miller \succ Brown \succ Smith.$$

When trying to apply the Choquet integral to represent the current manager's preferences, we realize that it is not possible. Indeed, by computing the Choquet integral of the performances of the candidates *Brown* and *Smith* we get

$$6 \cdot \mu(\{Ex, Ag\}) + (10 - 6) \cdot \mu(\{Ex\}) > 6 \cdot \mu(\{Ex, Ag\}) + (10 - 6) \cdot \mu(\{Ag\}) \tag{3.35}$$

while comparing *Baker* and *Miller* we get

$$7 \cdot \mu(\{Ex, Ag\}) + (9 - 7) \cdot \mu(\{Ag\}) > 7 \cdot \mu(\{Ex, Ag\}) + (9 - 7) \cdot \mu(\{Ex\}). \tag{3.36}$$

From Eq. (3.35) we get $\mu(\{Ex\}) > \mu(\{Ag\})$ while from Eq. (3.36) we get $\mu(\{Ag\}) > \mu(\{Ex\})$, which are of course incompatible.

Observe, however, that if you rescale the criteria experience and age multiplying the relative performances of the candidates by 0.56 and 0.44, respectively, we get the performances shown in Table 3.30. Computing the Choquet integral of the five candidates considering the capacity previously defined ($\mu(\{Ex\}) = 0.4$, $\mu(\{Ag\}) = 0.3$ and $\mu(\{Ex, Ag\}) = 1$), we get the values in the last column of Table 3.30 which represent the preferences of the manager. □

Table 3.30: The five candidates evaluated on experience and age after rescaling

	Ex	Ag	CI
Smith	3.36	4.4	3.67
Johnson	4.48	3.52	3.90
Brown	5.6	2.64	3.82
Baker	3.92	3.96	3.93
Miller	5.04	3.08	3.86

Let us conclude this section discussing the use of Choquet integral in case the DM prefers smaller objectives function values, i.e. for all $a, b \in A$, if $f_j(a) \leq f_j(b)$, then a is at least as good as b with respect to objective f_j. In this case, the objective functions f_j can still be aggregated using the Choquet integral, but then solutions with a smaller Choquet integral would be preferred, i.e., for all $a, b \in A$ if $C_\mu(a) \leq C_\mu(b)$, then a is comprehensively at least as good as b. In this case, the aim is to minimize rather than to maximize the value of the Choquet integral.

3.3.5 The NEMO-II-Ch method

Because most benchmark problems in evolutionary multiobjective optimization are *minimization* problems, in the following description of NEMO-II and of the empirical analysis, we assume that

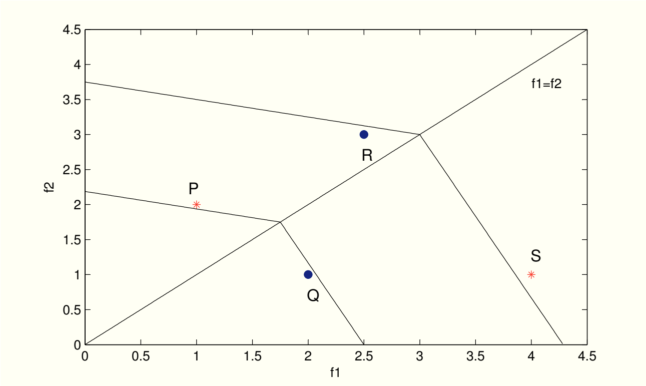


Figure 3.4: In this case, $Q \equiv (2,1)$ is preferred to $P \equiv (1,2)$ and $R \equiv (2.5,3)$ is preferred to $S \equiv (4,1)$. Trying to translate these preferences by using a linear model (3.28), we get the contradictory inequalities $w_1 < w_2$ and $w_1 > \frac{4}{3}w_2$. By using the Choquet integral preference model (3.31) to translate these preferences, we get the inequalities $m(\{1\}) < m(\{2\})$ and $m(\{1\}) > \frac{4}{3}m(\{2\}) + m(\{1,2\})$ being not in contradiction. For example, by considering $m(\{1\}) = 0.7$, $m(\{2\}) = 0.8$ and $m(\{1,2\}) = -0.5$ these inequalities are satisfied.

objective functions as well as a supposed users' utility are to be minimized. Compared to the discussion above, the only difference is that while with maximization, the preference of an alternative a over an alternative b was translated into the constraint $U(a) \geq U(b) + \varepsilon$, now the preference is translated into the constraint $U(a) + \varepsilon \leq U(b)$. In case of indifference between a and b, the corresponding constraint is $U(a) = U(b)$.

Let us first consider a problem with two objectives f_1 and f_2 to be minimized. As observed in Section 3.3.4, due to some interactions between the considered criteria it could happen (as in the case shown in Figure 3.4), that the linear model is not able to represent the preferences of the DM. For this reason, we suggest using the Choquet integral preference model that is able to take into account interactions between criteria. In [136], it has been shown experimentally that the Choquet integral has a greater capacity of representing the preferences of a DM than the weighted sum model.

While a greater flexibility of the preference model allows to capture more complicated user preference information and is thus desirable, usually, it also has more parameters, and more preference information is required before the set of compatible value functions is curbed sufficiently to be useful in narrowing down the search. For this reason, we propose to keep the complexity of the preference model low as long as it is sufficient to capture all preference information, but switch to a more complex preference model when this is no longer the case, following the procedure described in Algorithm 2. In particular, we start with assuming a linear preference model. Once we can no longer find a linear value function compatible with all elicited preference relations, we switch to a 2-additive

Algorithm 2 Basic NEMO-II-Ch method

Current preference model = LINEAR.
Generate initial population of solutions.
Elicit user's preferences by asking DM to compare two randomly selected non-dominated solutions.

Rank individuals into fronts by iteratively identifying all solutions that are most preferred for at least one compatible value function. Rank within each front using crowding distance.
repeat
  Select individuals for mating.
  Generate offspring using crossover and mutation and add them to the population.
  if Time to ask the DM **then**
    Elicit user's preferences by asking DM to compare two solutions that are the most preferred for at least one compatible value function.
    if There is no value function remaining compatible with the user's preferences **then**
      if Current preference model = LINEAR **then**
        Preference model = CHOQUET.
      else
        Remove information on pairwise comparisons, starting from the oldest one, until feasibility is restored and reintroduce them in the reverse order as long as feasibility is maintained.
      end if
    end if
  end if
  Rank individuals into fronts by iteratively identifying all solutions that are most preferred for at least one compatible value function. Rank within each front using crowding distance.
  Reduce population size back to initial size by removing worst individuals.
until Stopping criterion met.

Choquet integral.

To check whether there exists a set of weights $\mathbf{w} = (w_1, \ldots, w_n)$ such that the linear model is able to restore the preferences of the DM, one has to solve the following LP,

$$\max \varepsilon, \;\text{ subject to}$$
$$\left.\begin{array}{l} U(a) - U(b) + \varepsilon \leq 0 \;\text{ for all }\; a \succ_p b, \\ \displaystyle\sum_{j=1}^{n} w_j = 1 \\ w_j \geq 0, \;\text{for all } j = 1, \ldots, n \end{array}\right\} E_{DM}^{linear}$$

where $U(a) - U(b) \leq \varepsilon$ are the constraints translating the preferences of the DM while the other two constraints are used to ensure that weights are non-negative and normalized.

There is a $\mathbf{w}$ compatible with the preferences of the DM if and only if E_{DM}^{linear} is feasible and $\varepsilon^{linear} > 0$ where $\varepsilon^{linear} = \max \varepsilon$ subject to E_{DM}^{linear}. In this case, one can proceed to order the population by using the same procedure described in the previous section, checking, for each solution x, whether there exists a set of weights $\mathbf{w} = (w_1, \ldots, w_n)$ such that x is the best among the considered solutions. To this end, one has to solve the following LP.

$$\max \varepsilon, \;\text{ subject to}$$
$$\left.\begin{array}{l} U(a) - U(b) + \varepsilon \leq 0 \;\text{ for all }\; a \succ_p b, \\ U(x) - U(y) + \varepsilon \leq 0 \;\text{ for all } y \in A \setminus \{x\}, \\ \displaystyle\sum_{j=1}^{n} w_j = 1 \\ w_j \geq 0, \;\text{for all } j = 1, \ldots, n \end{array}\right\} E_{x}^{linear}$$

where constraints $U(x) - U(y) \leq \varepsilon$ ensure that x is preferred to all other solutions in A.

If E_x^{linear} is feasible and $\varepsilon_x^{linear} > 0$ where $\varepsilon_x^{linear} = \max \varepsilon$ subject to E_x^{linear}, then there exists a set of weights $\mathbf{w}$ such that x is the preferred solution and therefore it is included in the first front. After ordering all the solutions into different fronts, the solutions in the same front are ordered by computing the classical crowding distance of NSGA-II.

If there exists no vector $\mathbf{w}$ such that the linear model would be able to restore the preference information provided by the DM, we need to use a more complex model such as the Choquet integral in order to represent the preferences expressed by the DM. As we shall justify later, we use the 2-additive Choquet integral here that has a parameter $m(\{j\})$ for each objective j and a parameter

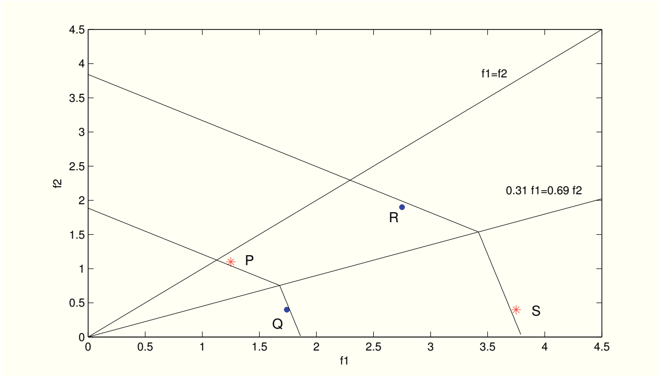


Figure 3.5: Example for a case where for capturing a user's preferences with the Choquet integral it is necessary to move the line $f_2 = f_1$ by multiplying both objectives by weights w_1 and w_2 such that $w_1 + w_2 = 1$.

$m(\{i,j\})$ for each pair of objectives $\{i,j\}$.

As already discussed in Section 3.3.4, the use of the Choquet integral assumes that all objectives are expressed on the same scale. Indeed, when using the original formulation of the Choquet integral involving the capacity μ (see Eq. (3.29)), for each solution x we need to order all values of each individual's objective from the worst to the best. Analogously, using the Choquet integral expressed by means of the Möbius decomposition m (see Eq. (3.31)), for each solution x and for each subset of objective functions, we need to know the minimum value. Since we cannot assume that the scales of the different objectives are comparable, the scaling becomes part of the model. In other words, in addition to the usual parameters of the Choquet integral, we also need to consider a scaling weight for each objective.

Let us consider the example shown in Figure 3.5. If the user states that $Q \equiv (1.75, 0.4)$ is preferred to $P \equiv (1.25, 1.05)$ and $R \equiv (2.75, 1.9)$ is preferred to $S \equiv (3.75, 0.4)$, there is no Choquet integral compatible with these preferences. The preferences correspond to the inequalities $0.5m(\{1\}) - 0.65m(\{2\}) - 0.65m(\{1,2\}) < 0$ and $m(\{1\}) - 1.5m(\{2\}) - 1.5m(\{1,2\}) > 0$ which contradict the monotonicity constraints 2c). This is also apparent from the fact that all four alternatives are located under the line $f_1 = f_2$, where the iso-utility function of the Choquet integral is linear, and no linear model is able to reflect the preferences.

However, if we scale the objectives appropriately (which "moves" the $f_1 = f_2$ line), then it is possible to represent the preference information using a Choquet integral. For example, let us multiply the two objectives by the weights $w_1 = 0.31$ and $w_2 = 0.69$. The four points become $Q \equiv (0.5425, 0.276)$, $P \equiv (0.3875, 0.7245)$, $R \equiv (0.8525, 1.311)$ and $S \equiv (1.1625, 0.276)$ while the constraints translating the preferences of the user become $0.155m(\{1\}) - 0.4485m(\{2\}) - 0.1115m(\{1,2\}) < 0$ and $0.31m(\{1\}) - 1.035m(\{2\}) - 0.5765m(\{1,2\}) > 0$, being compatible with the Möbius decomposition $m(\{1\}) = 0.9$, $m(\{2\}) = 0.4$ and $m(\{1,2\}) = -0.3$.

Mathematically, determining whether the preferences can be represented by an appropriate scaling and Choquet integral translates into the following non-linear program:

$$\max \varepsilon, \ \text{subject to}$$
$$\left.\begin{array}{l} C_\mu(w_1 f_1(a), \ldots, w_n f_n(a)) \\ \quad -C_\mu(w_1 f_1(b), \ldots, w_n f_n(b)) \leq \varepsilon \ \text{ for all } \ a \succ_p b, \\ \sum_{j=1}^{n} w_j = 1 \\ m(\emptyset) = 0, \\ \sum_{j=1}^{n} m(\{j\}) + \sum_{\{i,j\} \subseteq \{1,\ldots,n\}} m(\{i,j\}) = 1, \\ m(\{j\}) \geq 0, \ \text{for all } j = 1, \ldots, n, \\ m(\{j\}) + \sum_{i \in T} m(\{i,j\}) \geq 0, \ \text{for all } j = 1, \ldots, n, \ \text{and} \\ \quad \text{for all } T \subseteq \{1, \ldots, n\} \setminus \{j\}, \, T \neq \emptyset. \end{array}\right\} E_{DM}^{Ch} \tag{3.37}$$

The user's preferences can be represented if an only if the solution to this optimization problem results in $\varepsilon_{DM}^{Ch} > 0$, where $\varepsilon_{DM}^{Ch} = \max \varepsilon$ subject to E_{DM}^{Ch}.

Since the optimization problem is non-linear, to solve it, we use the Nelder-Mead algorithm [131] to search the space of weights while maximizing ε, with an LP being solved in every iteration to determine the best Möbius parameters for the weights in the current iteration. The algorithm is aborted as soon as an $\varepsilon > 0$ has been found. If after a few iterations (we chose 40 in the experiments below), no such weight/Choquet coefficient combination has been found, we stop the search and remove some of the DM's preference information, starting from the oldest, until the feasibility is regained.

Once it has been found that the preference information provided by the DM can be represented by using the Choquet integral preference model (possibly after removing some pieces of information), the solutions are put into different fronts by using the same procedure as described previously. For each solution $x \in A$, one has to check whether this solution might be the most preferred one by solving the following optimization problem.

$$\left.\begin{array}{l}\max \varepsilon, \text{ subject to} \\ C_\mu(w_1 f_1(x), \ldots, w_n f_n(x)) \\ -C_\mu(w_1 f_1(y), \ldots, w_n f_n(y)) \leq \varepsilon, \text{ for all } y \in A \setminus \{x\}, \\ E^{Ch}_{DM}\end{array}\right\} E^{Ch}_x \tag{3.38}$$

Because, again, the optimization problem (3.38) is non-linear, in a first step we try to simplify the solution by fixing the vector of weights $\mathbf{w}' = (w'_1, \ldots, w'_n)$ such that $w'_1 f_1(x) = \ldots = w'_n f_n(x)$ and then checking, by using linear programming optimization, if there exists a set of Choquet coefficients such that $\varepsilon^{Ch}_x > 0$, where $\varepsilon^{Ch}_x = \max \varepsilon$ s.t. E^{Ch}_x (indeed, if we fix the weights $(w_1, \ldots, w_n)$, then the optimization problem (3.38) becomes an LP problem). We found that in many cases, this will find a feasible solution if there exists one. If not, then we use the Nelder-Mead method explained above to check whether the optimization problem allows for $\varepsilon^{Ch}_x > 0$.

Note that for more than 2 dimensions, we still restrict our model to 2-additive Choquet.

3.3.6 Experimental Results

In this section, the algorithms introduced before are compared empirically. We start with a comparison of NEMO-I and NEMO-II to justify our use of NEMO-II for the remainder of the paper. Next, we look at the effect of model complexity, and demonstrate the benefit of starting with a simple model but then switching to a more complex model if it is necessary to represent the user's preferences. Finally, we compare a number of algorithms on various 2 to 5-dimensional benchmark functions.

For the algorithms tested, we use the following notation:

- **NEMO-I-L:** The NEMO-I algorithm with a *linear* additive preference model.
- **NEMO-II-L:** The NEMO-II algorithm with a *linear* additive preference model.
- **NEMO-II-PL2:** The NEMO-II algorithm with a piecewise linear additive preference model, consisting of two linear pieces. The breakpoint was chosen to be the median of the values in the population for each objective.
- **NEMO-II-G:** The NEMO-II algorithm with a general monotonic additive preference model as often used in ROR [86].

- **NEMO-II-Ch:** The proposed NEMO-II algorithm that starts with a simple linear preference model and switches to a 2-additive Choquet preference model when the linear model is no longer able to account for the user's preference information.

Furthermore, the following are provided as benchmarks:

- **NSGA-II:** The standard NSGA-II algorithm not using preference information.
- **EA-UVF:** A single-objective EA that uses the true user's value function for ranking individuals (information that is not available to the other algorithms). EA-UVF shows the performance that could be expected if the user's value function was fully known to the evolutionary algorithm from the beginning. Clearly, this is an idealized setting and only serves as a reference.
- **Optimum:** This is the best value of feasible solutions according to the true user's value function.

Let us point out that comparison between different interactive methods is difficult since they use different preference information. Consequently, we decided to compare our method with NSGA-II only because NEMO-II-Ch is based on NSGA-II.

All algorithms use a real valued representation, generate offspring by simulated binary crossover with crossover probability of 0.9 and $\eta_c = 15$, and Gaussian mutation with mutation probability $\frac{1}{v}$ (where v is the number of variables and depends on the considered problem) and step size $\sigma = 0.1$. Mating selection is done by tournament selection. We run the algorithm for a pre-specified number of 400 (in case of 2 objectives) or 600 (in case of 3 or 5 objectives) generations. The population size has been set to 30, a value smaller than usual in MOEAs, but we do not aim to find the whole Pareto frontier but only the most preferred solutions. The DM is asked to provide some preference information about one pair of randomly picked non-dominated solutions every 10 generations.

The "true" user's value function assumed in this study is the Chebyshev function, i.e., the user's goal is to maximize $U_{DM}(a) = -\max\{w_1 f_1(a), \ldots, w_n f_n(a)\}$, which is equivalent to minimizing $U_{DM}^{-}(a) = \max\{w_1 f_1(a), \ldots, w_n f_n(a)\}$. The parameters $w_1, \ldots, w_n$ depend on the problem and are defined below. For the sake of simplicity, with a slight abuse of the terminology, when we speak of the user's value functions we refer to their opposite forms, and thus we aim at minimizing the Chebyshev-like value function $U_{DM}^{-}(a) = \max\{w_1 f_1(a), \ldots, w_n f_n(a)\}$. When showing the plots of convergence and tables, the terms "Convergence indicator" and "Convergence curve" mean the value of $U_{DM}^{-}(\cdot)$.

We measure performance based on the user's true utility of the best individual in the population, i.e., the minimum of $U(x)$ over all individuals in the population. This assumes that the final population is returned to the DM, and the DM is able and willing to spend the effort to identify the most preferred solution among the set. All results shown in all tables and figures have been averaged over 50 independent runs.

Comparison of NEMO-I and NEMO-II

Since we implement here for the first time an idea that has been proposed in [24] as NEMO-II, we would like to assert that it is not only much more efficient than NEMO-I, but also competitive in terms of solution quality. To do this, we have compared NEMO-I and NEMO-II with a simple linear preference model on the simple convex 2-dimensional test problem ZDT1. The user's true value function was set to $U(x) = f_1(x) + f_2(x)$, i.e., the linear model is sufficient to represent the user's preferences. Knowing the user's true value function, one can calculate the solution with the best true user value for a given problem. This best value will be called "optimum". Figure 3.6 shows the convergence curve corresponding to this experiment. One can observe that both methods converge to the optimum ($f_1(x) = \frac{1}{4}$ and $f_2(x) = \frac{1}{2}$, with $U(x) = \frac{3}{4}$), and that the convergence of NEMO-II-L is slightly quicker than that of NEMO-I-L. We thus conclude that NEMO-II is not only computationally much more efficient than NEMO-I, but at least as good, and we will focus on NEMO-II in the remainder of this paper.

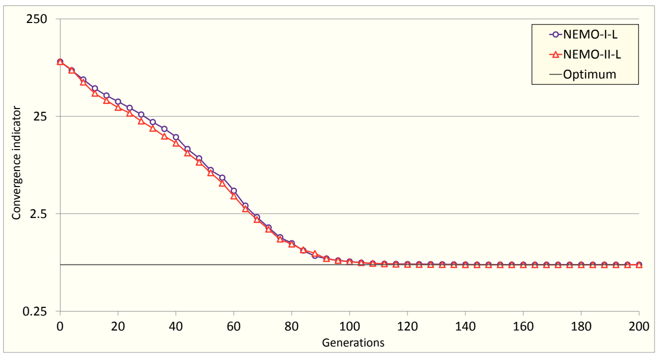


Figure 3.6: ZDT1-2D Linear ($f_1(x) + f_2(x)$)

Model complexity switching

Determining an appropriate model complexity is a challenging issue. Obviously, a higher complexity of the algorithm's preference model allows the algorithm to model more complex user's preferences.

On the other hand, higher model complexity usually means more parameters to be set, and thus more preference information is required for the model to become sufficiently restricted to be useful in guiding the evolutionary process. Besides, more parameters also means a higher computational cost for solving each LP in NEMO-II. For these reasons, we have proposed in Section 3.3.5 NEMO-II-Ch that starts with a simple linear preference model, and switches to a 2-additive Choquet integral when the simple linear model is no longer able to account for the user's preference information.

Figure 3.7 compares the proposed NEMO-II-Ch with NEMO-II-L, with NEMO-II that uses the 2-additive Choquet integral from the beginning, and with NEMO-II that starts using the linear model, but then switches to the full Choquet model rather than the 2-additive one. The test problem is the 5-dimensional DTLZ1, and the user preference function to be minimized is the Chebyshev function:

$$U(x) = \max \{0.1 f_1(x), 0.15 f_2(x), 0.2 f_3(x), 0.25 f_4(x), 0.3 f_5(x)\}. \tag{3.39}$$

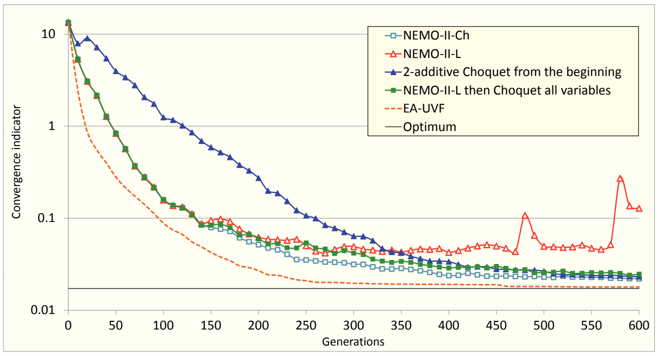


Figure 3.7: DTLZ1-5D Chebyshev with different types of Choquet integral as preference model.

As can be observed, the algorithm variants starting with a linear preference model converge much more quickly. This makes intuitive sense, as they have fewer parameters to estimate and thus can narrow down the search more quickly. Around iteration 70, the convergence curves for NEMO-II-L, NEMO-II-Ch and "NEMO-II-L then Choquet all variables" diverge, i.e., the linear model is sometimes no longer sufficient to capture the user's preference information, and switching to the Choquet model is required (note that not all runs switch at the same time). The difference between switching to the full Choquet integral or 2-additive Choquet integral is relatively small, but the full Choquet integral is somewhat slower, probably because it increases the number of parameters which slows down convergence. There doesn't seem to be a difference in final solution quality. Sticking to the linear model eventually results in poor and unstable behavior, as the model can no longer capture the user's preferences and old preference information has to be discarded.

In the following, we will thus stick to NEMO-II-Ch as our most promising procedure. To demonstrate the advantage of learning user's preferences, we compare it on various benchmark problems with NSGA-II. To further demonstrate the particular advantages of the Choquet integral over alternative value function models, we also report on results obtained with NEMO-II-L, NEMO-II-PL2, and NEMO-II-G.

Results in 2D

As 2D test problems, we use ZDT1 (with a convex Pareto front) and ZDT2 (with concave Pareto front) and the parameters of the Chebyshev function given in Table 3.31.

Table 3.31: Parameters of the user's value function in 2D

	w_1	w_2
ZDT1-middle	0.6	0.4
ZDT1-extreme	0.15	0.85
ZDT2-middle	0.6	0.4
ZDT2-extreme	0.15	0.85

As one can observe in Figures 3.8 and 3.9, the differences of final solution quality of the various approaches are very small, and all get very close to the optimal solution. This is not very surprising, since although the linear model can not represent the Chebyshev user preference directly, given that ZDT1 has a convex Pareto front, even the linear model is able to converge to any solution on the frontier. The fact that NEMO-II-Ch and NEMO-II-L lines overlap almost completely indicates that a linear model was almost always able to respect the user's preferences. Where the user prefers an extreme solution, the convergence of NSGA-II and NEMO-II-G is somewhat slower.

Moving to ZDT2, the situation is slightly different. NEMO-II-L is no longer able to converge to the correct point because of the concavity of the Pareto front of the test function as can be noticed from the erratic behavior of the corresponding curve in Figure 3.10 and especially in Figure 3.11. NSGA-II and NEMO-II-G again converge more slowly in the case of the user preferring an extreme solution.

For both two-dimensional test problems and both user's preference functions, the convergence curves for NSGA-II and NEMO-II-G are very similar, indicating that NEMO-II-G was not able to use the provided preference information to substantially enrich the standard non-dominated sorting

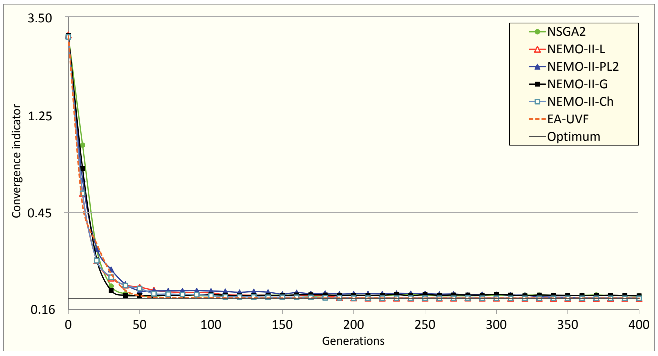


Figure 3.8: ZDT1-middle

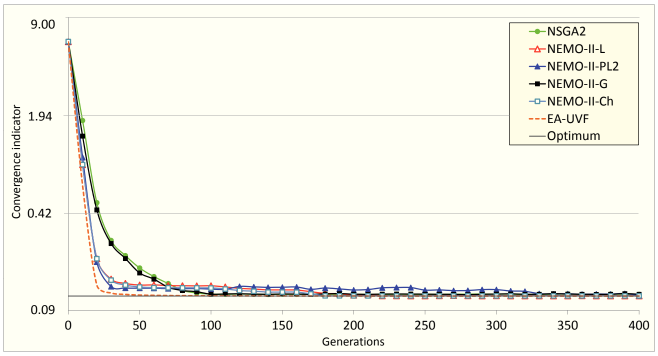


Figure 3.9: ZDT1-extreme

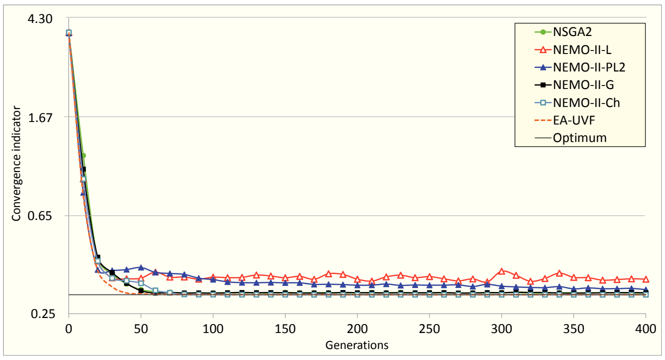


Figure 3.10: ZDT2-middle

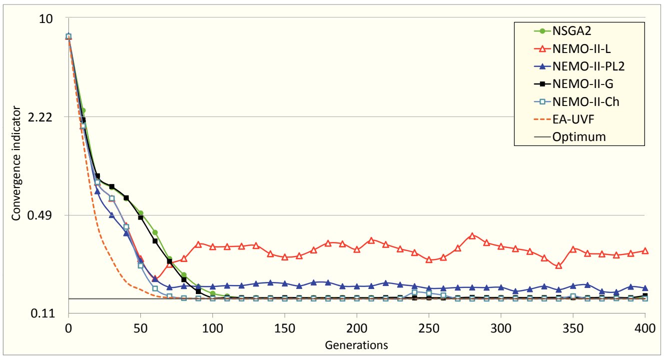


Figure 3.11: ZDT2-extreme

procedure.

Rather than comparing the value obtained after a specific number of generations, Table 3.32 shows the area under the convergence curve over all 400 generations as a measure of the overall performance of the algorithm. Based on a Mann-Whitney-U test with 5% significance level, the differences between NEMO-II-Ch and each of the other tested algorithms are significant, except for ZDT1 and the difference to NEMO-II-L. The latter is not surprising, given that for such a convex problem a linear model is able to find any solution on the frontier.

To illustrate the convergence of NEMO-II-Ch in the objective space, we show in Figure 3.12 the population of solutions of ZDT2-middle after 30, 50, 100 and 200 generations. One can observe that after 30 generations the population of solutions is located in the upper left corner of the Pareto front which corresponds to one of the two minima obtained by a linear value function model. Then, in subsequent generations the method discovers that the linear model is not able to represent the growing set of pairwise comparisons and switches to the Choquet integral model. From generation 50 on, the method generates solutions grouped around the point most preferred by the artificial user. In generations 100 and then 200, the population is almost exactly focused on this point.

Table 3.32: Area under the curve for 2-dimensional problems, mean ± std. err. The difference of all results to the results of NEMO-II-Ch is significant except for NEMO-II-L in ZDT1.

	ZDT1	
	middle	extreme
NSGA-II	141.38 ± 1.88	200.03 ± 4.12
NEMO-II-L	127.32 ± 1.76	164.48 ± 4.09
NEMO-II-PL2	138.27 ± 1.51	180.14 ± 3.04
NEMO-II-G	139.38 ± 0.94	196.22 ± 2.57
NEMO-II-Ch	**126.73** ± 1.62	**163.73** ± 4.11
	ZDT2	
	middle	extreme
NSGA-II	199.96 ± 2.02	255.08 ± 4.67
NEMO-II-L	207.80 ± 1.95	275.34 ± 5.24
NEMO-II-PL2	205.17 ± 1.40	254.36 ± 3.79
NEMO-II-G	199.99 ± 1.08	252.38 ± 2.48
NEMO-II-Ch	**189.55** ± 1.61	**225.27** ± 4.56

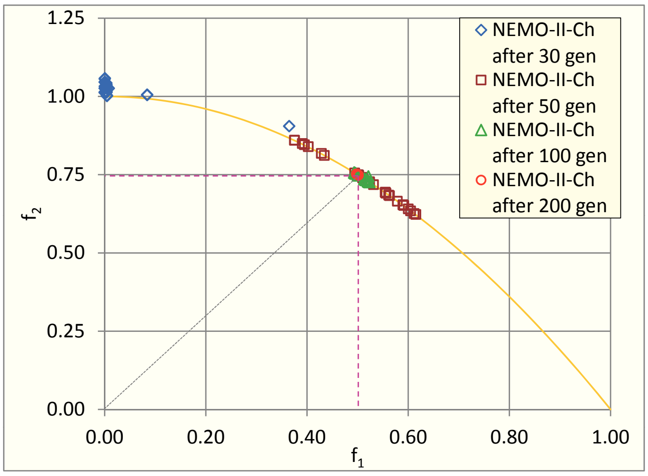


Figure 3.12: Convergence to the best preferred solution by NEMO-II-Ch for ZDT2-middle

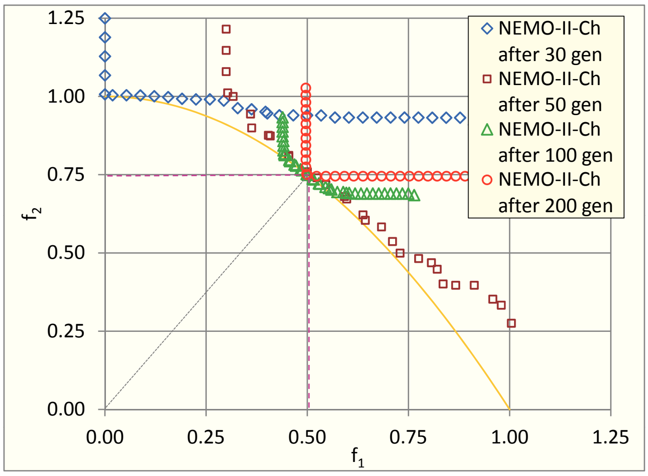


Figure 3.13: Attainment surfaces by NEMO-II-Ch for ZDT2-middle

It is also interesting to observe the convergence process using the attainment surface plots [111] for the same problem. They are shown in Figure 3.13. The median attainment surface shows the area that is dominated by solutions from the population of a given generation in 50% of the runs. In Figure 3.13 the dominated area is located on the upper right side of the curves. The obtained surface after 30 generations has almost conic shape because the population of solutions is concentrated in the left upper corner of the Pareto front. After switching from the linear to the Choquet integral model the attainment surfaces become more and more conic and, finally, in generation 200 it is perfectly orthogonal. This means that the whole population is focused on one, most preferred point.

Results in 3D

In three dimensions we have compared the five methods on the benchmark problems DTLZ1 and DTLZ2 considering the user's Chebyshev value function with the parameters given in Table 3.33.

Table 3.33: Parameters of the user's value function in 3D

	w_1	w_2	w_3
DTLZ1-3D-middle	0.3	0.4	0.3
DTLZ1-3D-extreme	0.2	0.3	0.5
DTLZ2-3D-middle	0.3	0.4	0.3
DTLZ2-3D-extreme	0.2	0.3	0.5

What we can observe for DTLZ1 in Figures 3.14 and 3.15 is that only NEMO-II-Ch is able to get very close to the optimal solution. Indeed, after generation 200 the curve of NEMO-II-Ch merges with that of EA-UVF, which means the NEMO-II-Ch behaves like a single-objective EA using the true user's value function. NSGA-II is the second closest, but converges significantly slower. NEMO-II-L and NEMO-II-PL2 show erratic behavior, and NEMO-II-G converges very slowly, apparently unable to make use of the provided preference information. Note that the peaks in the plots of NEMO-II-L and NEMO-II-PL2 appear when in order to get a compatible model some of the oldest preference information has to be removed. This removal deteriorates temporarily the value of the best solution in the current population.

From Figures 3.16 and 3.17, one can notice that on DTLZ2, NEMO-II-Ch is slower than NSGA-II or even NEMO-II-G in the beginning, although it takes over eventually and finds better solutions. We are not sure what characteristics of DTLZ2 cause this behavior. Something seems to mislead the linear model in the wrong direction, which is doing really poorly overall. However, NEMO-II-Ch is

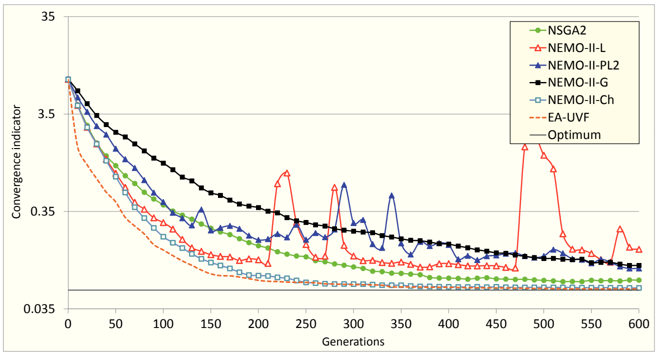


Figure 3.14: DTLZ1-3D-middle

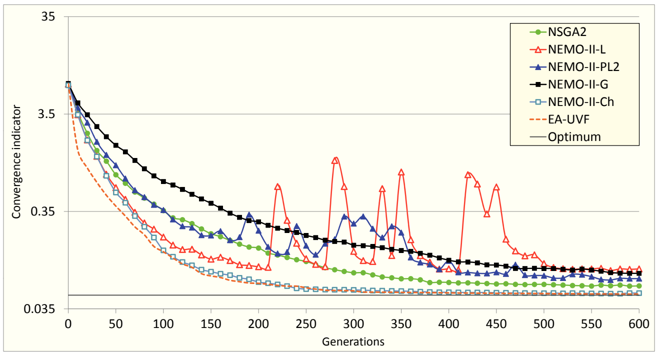


Figure 3.15: DTLZ1-3D-extreme

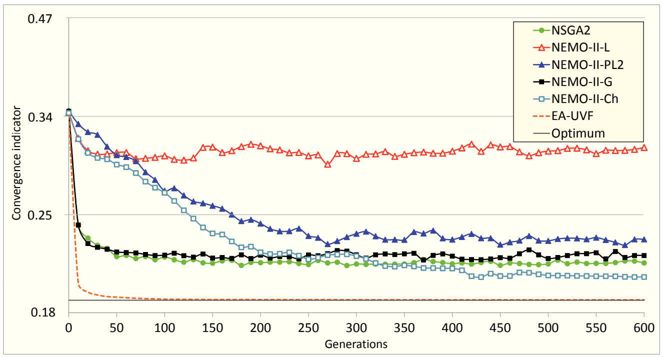


Figure 3.16: DTLZ2-3D-middle

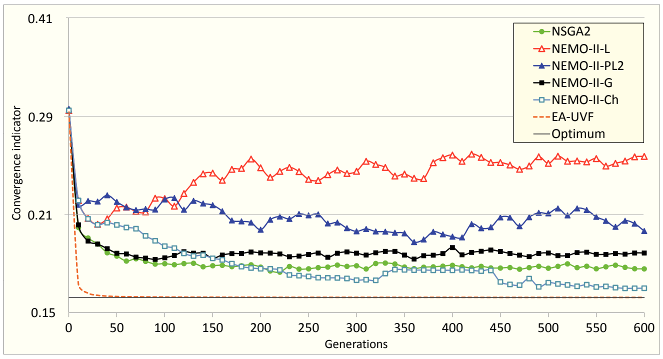


Figure 3.17: DTLZ2-3D-extreme

Table 3.34: Area under the curve for 3-dimensional problems, mean ± std. err. The following differences to NEMO-II-CH are not significant based on a Mann-Whitney-U test with 5% significance level: DTLZ2 extreme and NSGA-II, DTLZ2 middle and NEMO-II-G.

	DTLZ1	
	middle	extreme
NSGA-II	334.78± 9.64	291.60 ± 7.93
NEMO-II-L	391.65± 41.33	326.68 ± 27.10
NEMO-II-PL2	446.30±12.74	361.94 ± 9.14
NEMO-II-G	562.98 ± 13.34	439.64 ± 10.40
NEMO-II-Ch	**288.17** ± 11.89	**244.99** ± 8.99
	DTLZ2	
	middle	extreme
NSGA-II	**128.98** ± 0.09	107.01 ± 0.10
NEMO-II-L	182.87 ± 2.01	144.03 ± 6.04
NEMO-II-PL2	148.71 ± 0.64	124.69 ± 0.78
NEMO-II-G	131.58 ± 0.07	111.24 ± 0.08
NEMO-II-Ch	137.96 ± 2.31	**106.90** ± 1.33

able to recover from a bad start when switching to the Choquet integral, and eventually yields the best results.

As the numerical results in Table 3.34 confirm, overall, NEMO-II-Ch is still best in three out of the four scenarios.

Results in 5D

In 5D we have considered the DTLZ1 and DTLZ2 benchmark functions for two user's Chebyshev value functions with parameters given in Table 3.35.

Table 3.35: Parameters of the user's value function in 5D

	w_1	w_2	w_3	w_4	w_5
DTLZ1-5D-extreme$_1$	0.1	0.15	0.2	0.25	0.3
DTLZ1-5D-extreme$_2$	0.3	0.25	0.2	0.15	0.1
DTLZ2-5D-extreme$_1$	0.1	0.15	0.2	0.25	0.3
DTLZ2-5D-extreme$_2$	0.3	0.25	0.2	0.15	0.1

Different from the 2D and 3D cases, in which the performance of NSGA-II was good, in the 5D case, NSGA-II performs quite poorly. This is not very surprising, as it is known that the non-dominance ranking does not work effectively in more than 3 dimensions. In such cases, preference information is hugely beneficial as it allows to substantially enrich the non-dominated ranking. Figures 3.18-3.21 thus show that NEMO-II-Ch obtains much better results than any of the other algorithms on both benchmark problems and for both user preference functions. The two linear

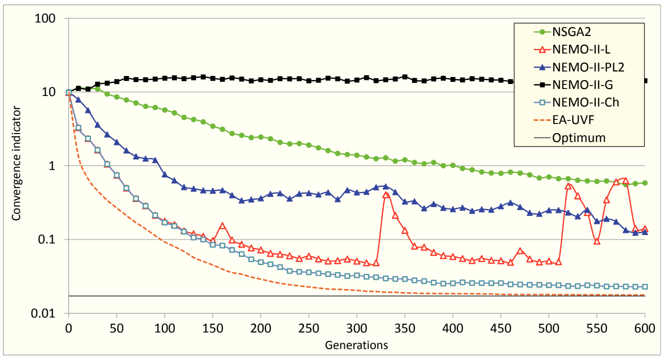


Figure 3.18: DTLZ1-5D-extreme$_1$

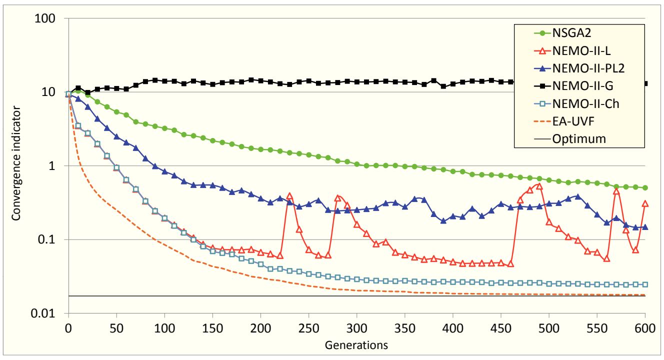


Figure 3.19: DTLZ1-5D-extreme$_2$

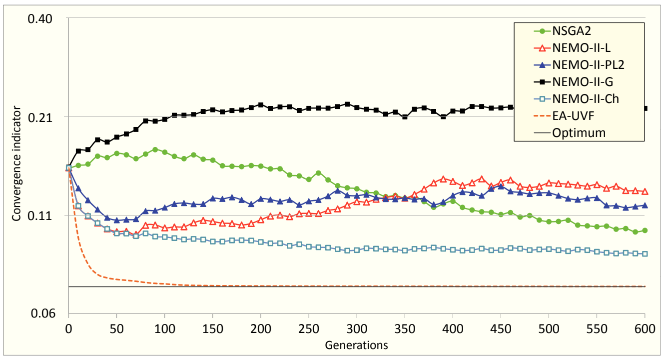


Figure 3.20: DTLZ2-5D-extreme$_1$

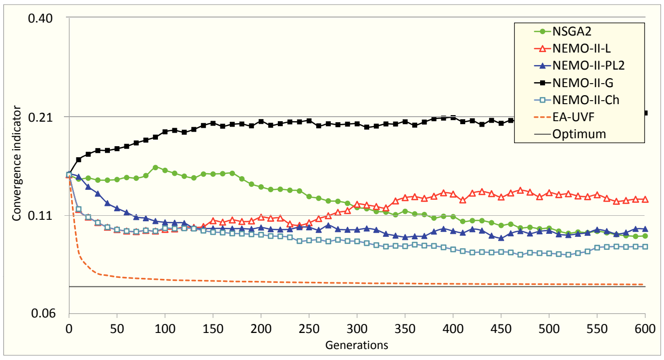


Figure 3.21: DTLZ2-5D-extreme$_2$

Table 3.36: Area under the curve for 5-dimensional problems, mean ± std. err. NEMO-II-Ch is significantly better than each other algorithm except for the comparison with NEMO-II-L on DTLZ1.

	DTLZ1	
	extreme$_1$	extreme$_2$
NSGA-II	1747.43 ± 70.29	1271.34 ± 50.33
NEMO-II-L	251.67 ± 19.80	250.81 ± 18.19
NEMO-II-PL2	586.27 ± 22.49	299.10 ± 21.89
NEMO-II-G	8569.95 ± 40.16	7531.91 ± 38.14
NEMO-II-Ch	**207.84** ± 9.75	**208.96** ± 9.25
	DTLZ2	
	extreme$_1$	extreme$_2$
NSGA-II	81.86 ± 1.34	74.20 ± 1.03
NEMO-II-L	73.47 ± 2.50	70.45 ± 1.95
NEMO-II-PL2	74.97 ± 0.51	63.59 ± 0.54
NEMO-II-G	129.65 ± 0.67	119.30 ± 0.71
NEMO-II-Ch	**57.02** ± 0.65	**58.06** ± 0.74

models NEMO-II-L and NEMO-II-PL2 again show an erratic behavior, presumably because they are not sufficiently complex to reflect the user's preference information. A bit surprising is the very poor behavior of NEMO-II-G. Apparently, in 5D, the general monotonic additive preference model is not helpful. Again, when looking at the numerical results in Table 3.36, we see that NEMO-II-Ch yields better results than all other algorithms in all cases.

Average utility

Finally, Table 3.37 looks at the *average* utility of all the individuals in the final population. This measure provides some information on how well the algorithm was able to focus the search onto the most preferred region of the search space, as keeping some individuals with poor utility in the population would hurt this performance measure.

Again, NEMO-II-Ch performs best for most scenarios. It is marginally worse than NEMO-II-L and NEMO-II-PL2 for ZDT1-middle where a linear value function model is sufficient to capture the user's preferences. And it is slightly worse than NEMO-II-PL2 on the 5-dimensional DTLZ2-extreme$_2$ problem, despite the fact that according to Figure 3.21, the best solution in the population of NEMO-II-PL2 is clearly worse than the best solution in the population of NEMO-II-Ch. The likely explanation is that although NEMO-II-PL2 converged to an inferior solution, it converged fully, whereas NEMO-II-Ch still had some diversity in the population that degraded the average utility.

Table 3.37: Average and stdv of the utilities for the individuals in the last population for all considered methods

	ZDT1-middle	ZDT1-extreme	ZDT2-middle	ZDT2-extreme
NSGA-II	0.3415 ± 0.117	0.3756 ± 0.2233	0.4247 ± 0.0868	0.5079 ± 0.2487
NEMO-II-L	$\mathbf{0.1825 \pm 0.0082}$	0.1182 ± 0.0643	0.4077 ± 0.1033	0.4304 ± 0.3214
NEMO-II-PL2	0.1853 ± 0.0155	0.1192 ± 0.0138	0.3364 ± 0.0374	0.3581 ± 0.1952
NEMO-II-G	0.3458 ± 0.0136	0.3740 ± 0.0299	0.4301 ± 0.0278	0.5298 ± 0.0519
NEMO-II-Ch	0.1856 ± 0.0262	$\mathbf{0.1169 \pm 0.0071}$	$\mathbf{0.3010 \pm 0.0106}$	$\mathbf{0.1418 \pm 0.0406}$

	DTLZ1-3D-middle	DTLZ1-3D-extreme	DTLZ2-3D-middle	DTLZ2-3D-extreme
NSGA-II	0.1252 ± 0.0373	0.129 ± 0.0545	0.2886 ± 0.049	0.3131 ± 0.1041
NEMO-II-L	3.5691 ± 16.8329	2.7865 ± 12.9957	0.3162 ± 0.0411	0.2812 ± 0.1038
NEMO-II-PL2	0.1237 ± 0.0521	0.1211 ± 0.0313	0.2454 ± 0.0404	0.2260 ± 0.0450
NEMO-II-G	0.1411 ± 0.0284	0.1384 ± 0.0098	0.3054 ± 0.0085	0.3324 ± 0.0113
NEMO-II-Ch	$\mathbf{0.0635 \pm 0.0197}$	$\mathbf{0.0541 \pm 0.0102}$	$\mathbf{0.2078 \pm 0.0395}$	$\mathbf{0.1705 \pm 0.0269}$

	DTLZ1-5D-extreme$_1$	DTLZ1-5D-extreme$_2$	DTLZ2-5D-extreme$_1$	DTLZ2-5D-extreme$_2$
NSGA-II	1.4225 ± 1.0363	1.9122 ± 1.7195	0.2168 ± 0.0864	0.1977 ± 0.0812
NEMO-II-L	3.0879 ± 13.0158	2.2674 ± 10.7805	0.1622 ± 0.0657	0.1381 ± 0.0465
NEMO-II-PL2	0.1922 ± 0.1720	0.1623 ± 0.3116	0.1491 ± 0.0468	$\mathbf{0.1214} \pm 0.0374$
NEMO-II-G	16.754 ± 4.772	17.494 ± 4.815	0.3806 ± 0.0620	0.3517 ± 0.0509
NEMO-II-Ch	$\mathbf{0.0368 \pm 0.0209}$	$\mathbf{0.0417 \pm 0.0205}$	$\mathbf{0.1449 \pm 0.0666}$	0.1353 ± 0.0687

3.3.7 Conclusions

In this paper, we presented the NEMO-II-Ch method. NEMO-II-Ch is an interactive evolutionary multiobjective procedure guided by user's preferences towards the most preferred part of the Pareto-optimal set. The novelties brought by the method consist in the following features:

- It is the first implementation and empirical evaluation of the NEMO-II idea.
- It does not work by considering only one model to translate the preferences of the DM but it starts from the simplest one (the linear model) and passes to a more complex one (the 2-additive Choquet integral model) when it is not possible to represent the DM's preferences using the linear model.
- The use of the Choquet integral preference model has never been considered in the evolutionary multiobjective optimization field for its relative complexity, however, we have demonstrated that it is able to deal efficiently with problems where preferences involve interactions among criteria, which additive preference models are unable to represent.

In order to demonstrate the effectiveness of the presented method, as well as the quality of its solutions, we have compared NEMO-II-Ch with NSGA-II and a variant of Greenwood's method (NEMO-II-L) on a variety of benchmark problems in 2D, 3D and 5D.

In almost all performed simulations, NEMO-II-Ch clearly obtained better results than the other tested methods.

Further developments will include to study how the increase of the number of interacting objectives in the k-additive Choquet integral for $k > 2$ influences the performance of the interactive procedure based on this preference model.

Chapter 4

Final remarks

The aim of the Multiple Criteria Decision Aiding (MCDA) is not addressing the Decision Maker (DM) to the "best" decision but, as stated by Keeney and Raiffa in [109], *...to get your head straightened out*. For this reason, several methods have been proposed during the years in MCDA and, in this thesis, we contributed in dealing with two important MCDA issues that are, the hierarchy of criteria and the interaction between criteria. Regarding the first issue, two contributions have been proposed:

- in the first contribution, we applied the recently proposed Multiple Criteria Hierarchy Process (MCHP) to extend the well-known sorting method UTADIS. The application of the MCHP to the UTADIS method permits to assign each alternative to one or more of the preferentially ordered classes not only at a comprehensive level but also considering a particular subcriterion in the hierarchy of criteria. This extension regards both the cases in which the DM provides direct and indirect preference information;

- in the second contribution, we applied the MCHP to the Choquet integral preference model to take into account positive or negative interactions between criteria structured in a hierarchy. Also in this case, the proposed extension regards both the direct and the indirect preference information. In case the DM provides indirect preference information, we applied the Robust Ordinal Regression (ROR) and the Stochastic Multiobjective Acceptability Analysis (SMAA) to take into account the plurality of models compatible with the preference information provided by the DM.

Regarding the second issue, that is the interaction between criteria, we gave three contributions related to the Choquet integral preference model:

- in the first contribution, we proposed a methodology to build a common scale, necessary for the application of the Choquet integral preference model. Moreover, to take into account the plurality of common scales that could be built by using the proposed method, we applied the SMAA methodology;

- in the second contribution we dealt, again, the problem of building a common scale for evaluations on different criteria but, differently from the previous contribution, we applied the Analytical Hierarchy Process (AHP) to build this common scale; moreover, we proposed a way to reduce the number of pairwise comparisons requested in the application of the AHP method;
- in the third contribution, we applied the Choquet integral preference model to the evolutionary multiobjective optimization. The Choquet integral preference model has been applied for the first time to this field and it is used to address the search to the region of the Pareto front most interesting for the DM. The proposed method, called NEMO-II-Ch, has been tested on several benchmark problems providing better results than the other methods with which it has been compared.

Bibliography

[1] `http://www.umultirank.org`.

[2] S. Angilella, M. Bottero, S. Corrente, V. Ferretti, S. Greco, and I. Lami. Non additive robust ordinal regression for urban and territorial planning: an application for siting an urban waste landfill. *Annals of Operations Research*, 2015. DOI:10.1007/s10479-015-1787-7.

[3] S. Angilella, S. Corrente, and S. Greco. SMAA-Choquet: Stochastic multicriteria acceptability analysis for the Choquet integral. In S. Greco et al., editor, *Advances in Computational Intelligence*, volume 300 of *Communications in Computer and Information Science*, pages 248–257. Springer Berlin Heidelberg, 2012.

[4] S. Angilella, S. Corrente, and S. Greco. Stochastic Multiobjective Acceptability Analysis for the Choquet integral preference model and the scale construction problem. *European Journal of Operational Research*, 240(1):172–182, 2015.

[5] S. Angilella, S. Corrente, S. Greco, and R. Słowiński. Multiple Criteria Hierarchy Process for the Choquet Integral. In R.C Purshouse, P.J. Fleming, C.M. Fonseca, S. Greco, and J. Shaw, editors, *Evolutionary Multi-Criterion Optimization*, volume 7811 of *Lecture Notes in Computer Science*, pages 475–489. Springer, Berlin, 2013.

[6] S. Angilella, S. Greco, F. Lamantia, and B. Matarazzo. Assessing non-additive utility for multicriteria decision aid. *European Journal of Operational Research*, 158(3):734–744, 2004.

[7] S. Angilella, S. Greco, and B. Matarazzo. The most representative utility function for non-additive robust ordinal regression. In E. Hullermeier, R. Kruse, and F. Hoffmann, editors, *Proceedings of IPMU 2010, LNAI 6178*, pages 220–229. Springer, Heidelberg, 2010.

[8] S. Angilella, S. Greco, and B. Matarazzo. Non-additive robust ordinal regression: A multiple criteria decision model based on the Choquet integral. *European Journal of Operational Research*, 201(1):277–288, 2010.

[9] C. Bana e Costa and J.-C. Vansnick. MACBETH-An interactive path towards the construction of cardinal value functions. *International transactions in operational Research*, 1(4):489–500, 1994.

[10] C. Bana e Costa and J.-C. Vansnick. A critical analysis of the eigenvalue method used to derive priorities in AHP. *European Journal of Operational Research*, 187(3):1422–1428, 2008.

[11] R. Battiti and A. Passerini. Brain-computer evolutionary multiobjective optimization: A genetic algorithm adapting to the decision maker. *IEEE Transactions on Evolutionary Computation*, 14(5):671–687, 2010.

[12] D.A. Bell, R.L Keeney, and H. Raiffa. *Conflicting objectives in decisions.* Wiley, 1977.

[13] D.E. Bell. Multiattribute utility functions: Decompositions using interpolation. *Management Science*, 25:744–753, 1979.

[14] V. Belton and T.J. Stewart. *Multiple criteria decision analysis: an integrated approach.* Springer, 2002.

[15] L. Berrah and V. Clivillé. Towards an aggregation performance measurement system model in a supply chain context. *Computers in Industry*, 58(7):709–719, 2007.

[16] S. Bortot and R.A. Marques Pereira. Inconsistency and non-additive capacities: The Analytic Hierarchy Process in the framework of Choquet integration. *Fuzzy Sets and Systems*, 213:6–26, 2013.

[17] C. Boutilier, F. Bacchus, and R.I. Brafman. UCP-networks: A directed graphical representation of conditional utilities. In *Proceedings of the Seventeenth conference on Uncertainty in artificial intelligence*, pages 56–64. Morgan Kaufmann Publishers Inc., 2001.

[18] D. Bouyssou and T. Marchant. Additive conjoint measurement with ordered categories. *European Journal of Operational Research*, 203(1):195–204, 2010.

[19] J. Branke. Consideration of user preferences in evolutionary multi-objective optimization. In J. Branke, K. Deb, K. Miettinen, and R.Słowiński, editors, *Multiobjective Optimization - Interactive and Evolutionary Approaches*, volume 5252 of *LNCS*, pages 157–178. Springer, 2008.

[20] J. Branke. MCDA and Multiobjective Evolutionary Algorithms. In S. Greco, M. Ehrgott, and J. Figueira, editors, *Mutiple criteria decision analysis: state of the art surveys*, chapter 22. Springer, 2nd edition, To appear.

[21] J. Branke, K. Deb, K. Miettinen, and R. Słowiński. *Multiobjective Optimization: Interactive and Evolutionary Approaches*, volume 5252. Springer, 2008.

[22] J. Branke, S. Greco, R. Słowiński, and P. Zielniewicz. Interactive evolutionary multiobjective optimization using robust ordinal regression. In M. Ehrgott et al., editors, *International Conference on Evolutionary Multi-Criterion Optimization*, volume 5467 of *LNCS*, pages 554–568. Springer, 2009.

[23] J. Branke, S. Greco, R. Słowiński, and P. Zielniewicz. Interactive evolutionary multiobjective optimization driven by robust ordinal regression. *Bulletin of the Polish Academy of Sciences - Technical Sciences*, 58(3):347–358, 2010.

[24] J. Branke, S. Greco, R. Słowiński, and P. Zielniewicz. Learning Value Functions in Interactive Evolutionary Multiobjective Optimization. *IEEE Transactions on Evolutionary Computation*, 19(1):88–102, 2015.

[25] J. Branke, T. Kaußler, and H. Schmeck. Guidance in evolutionary multi-objective optimization. *Advances in Engineering Software*, 32:499–507, 2001.

[26] J.P. Brans and B. Mareschal. PROMETHEE Methods. In J. Figueira, S. Greco, and M. Ehrgott, editors, *Multiple Criteria Decision Analysis: State of the Art Surveys*, pages 163–196. Springer, Berlin, 2005.

[27] J.P. Brans and Ph. Vincke. A preference ranking organisation method: The PROMETHEE method for MCDM. *Management Science*, 31(6):647–656, 1985.

[28] P. Campigotto and A. Passerini. Adapting to a realistic decision maker: experiments towards a reactive multi-objective optimizer. In *International Conference on Learning and Intelligent Optimization*, volume 6073 of *LNCS*, pages 338–341. Springer, 2010.

[29] A. Charnes, W.W. Cooper, and R.O. Ferguson. Optimal estimation of executive compensation by linear programming. *Management Science*, 1(2):138–151, 1955.

[30] A. Chateauneuf and J.Y. Jaffray. Some characterizations of lower probabilities and other monotone capacities through the use of Möbius inversion. *Mathematical Social Sciences*, 17:263–283, 1989.

[31] G. Choquet. Theory of capacities. *Annales de l'Institut Fourier*, 5:131–295, 1953.

[32] C.A. Coello Coello. Handling preferences in evolutionary multiobjective optimization: A survey. In *Congress on Evolutionary Computation*, volume 1, pages 30–37. IEEE, 2000.

[33] C.A. Coello Coello, D.A. Van Veldhuizen, and G.B. Lamont. *Evolutionary Algorithms for Solving Multi-Objective Problems*. Kluwer, 2002.

[34] S. Corrente, J.R. Figueira, and S. Greco. Dealing with interaction between bipolar multiple criteria preferences in PROMETHEE methods. *Annals of Operations Research*, 217(1):137–164, 2014.

[35] S. Corrente, S. Greco, M. Kadziński, and R. Słowiński. Robust Ordinal Regression in preference learning and ranking. *Machine Learning*, 93(2-3):381–422, 2013.

[36] S. Corrente, S. Greco, M. Kadziński, and R. Słowiński. Robust ordinal regression. *Wiley Encyclopedia of Operations Research and Management Science*, pages 1–10, 2014.

[37] S. Corrente, S. Greco, and R. Słowiński. Multiple Criteria Hierarchy Process in Robust Ordinal Regression. *Decision Support Systems*, 53(3):660–674, 2012.

[38] S. Corrente, S. Greco, and R. Słowiński. Multiple Criteria Hierarchy Process with ELECTRE and PROMETHEE. 41:820–846, 2012.

[39] K. Deb, S. Agrawal, A. Pratap, and T. Meyarivan. A fast elitist non-dominated sorting genetic algorithm for multi-objective optimization: NSGA-II. In *Parallel Problem Solving from Nature PPSN VI*, pages 849–858. Springer, 2000.

[40] K. Deb and S. Chaudhuri. I-MODE: An interactive multi-objective optimization and decision-making using evolutionary methods. Technical Report KanGAL Report No. 2007003, Indian Institute of Technology Kanpur, 2007.

[41] K. Deb, A. Sinha, P.J. Korhonen, and J. Wallenius. An interactive evolutionary multiobjective optimization method based on progressively approximated value functions. *IEEE Transactions on Evolutionary Computation*, 14(5):723–739, 2010.

[42] G. Debreu. Representation of a preference ordering by a numerical function. *Decision processes*, 3:159–165, 1954.

[43] J.M. Devaud, G. Groussaud, and E. Jacquet-Lagréze. Une méthode de construction de fonctions d'utilité additives rendant compte de judgments globaux. In *Proc. EURO Working Group Meeting on Multicriteria Decision Aiding*, Bochum, 1980.

[44] D. Diakoulaki, C.H. Antunes, and A. Gomes Matins. MCDA and energy planning. In J. Figueira, S. Greco, and M. Ehrgott, editors, *Multiple criteria decision analysis: State of the art surveys*, pages 859–897. Springer, Berlin, 2005.

[45] M. Doumpos and C. Zopounidis. *Multicriteria Decision Aid Classification Methods*. Springer, New York, 2002.

[46] M. Doumpos and C. Zopounidis. A multicriteria decision support system for bank rating. *Decision Support Systems*, 50(1):55–63, 2010.

[47] M. Doumpos and C. Zopounidis. *Multicriteria Analysis in Finance*. Springer, 2014.

[48] M. Doumpos, C. Zopounidis, and E. Galariotis. Inferring robust decision models in multicriteria classification problems: An experimental analysis. *European Journal of Operational Research*, 236:601–611, 2014.

[49] M. Elliott. Selecting numerical scales for pairwise comparisons. *Reliability Engineering and System Safety*, 95(7):750–763, 2010.

[50] P.H. Farquhar. A Fractional Hypercube Decomposition Theorem for Multiattribute Utility Functions. *Operations Research*, 23:941–967, 1975.

[51] J. Figueira, S. Greco, and M. Ehrgott (eds.). *Multiple Criteria Decision Analysis: State of the Art Surveys*. Springer, Berlin, 2005.

[52] J. Figueira, S. Greco, and B. Roy. ELECTRE methods with interaction between criteria: An extension of the concordance index. *European Journal of Operational Research*, 199(2):478–495, 2009.

[53] J. Figueira, S. Greco, and R. Słowiński. Identifying the "most representative" value function among all compatible value functions in the GRIP. *Proceedings of the 68th EURO Working Group on MCDA, Chania*, October 2008.

[54] J. Figueira, V. Mousseau, and B. Roy. ELECTRE methods. In J. Figueira, S. Greco, and M. Ehrgott, editors, *Multiple criteria decision analysis: State of the art surveys*, pages 133–153. Springer, Berlin, 2005.

[55] J.R. Figueira, S. Greco, R. Słowiński, and B. Roy. An overview of ELECTRE methods and their recent extensions. *Journal of Multi-Criteria Decision Analysis*, 20:61–85, 2013.

[56] P.C. Fishburn. *Utility Theory for Decision Making.* Wiley, New York, 1970.

[57] P.C. Fishburn and R.L. Keeney. Generalized utility independence and some implications. *Operations Research*, 23:928–940, 1975.

[58] C. M. Fonseca and P. J. Fleming. Genetic algorithms for multiobjective optimization: Formulation, discussion, and generalization. In *Proceedings of the Fifth International Conference on Genetic Algorithms*, pages 416–423, 1993.

[59] J. W. Fowler, E. S. Gel, M. M. Köksalan, P. Korhonen, J. L. Marquis, and J. Wallenius. Interactive evolutionary multi-objective optimization for quasi-concave preference functions. *European Journal of Operational Research*, 206:417–425, 2010.

[60] K. Fujimoto. On hierarchical decomposition of Choquet integral model. PhD thesis, Tokyo Institute of Technology, 1995.

[61] K. Fujimoto. Obtaining admissible preference orders using hierarchical bipolar Sugeno and Choquet integrals. *Journal of Advanced Computational Intelligence*, 17(4):493–513, 2013.

[62] K. Fujimoto and T. Murofushi. Hierarchical decomposition of the Choquet integral. In T. Murofushi M. Grabisch and M. Sugeno, editors, *Fuzzy Measures and Integrals: Theory and Applications*, pages 94–103. Physica-Verlag, 2000.

[63] K. Fujimoto, T. Murofushi, and M. Sugeno. Canonical hierarchical decomposition of Choquet integral over finite set with respect to null additive fuzzy measure. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 06(04):345–363, 1998.

[64] A. Giarlotta and S. Greco. Necessary and possible preference structures. *Journal of Mathematical Economics*, 49(2):163–172, 2013.

[65] G. Gigerenzer and W. Gaissmaier. Heuristic decision making. *Annual review of psychology*, 62:451–482, 2011.

[66] I. Gilboa and D. Schmeidler. Additive representations of non-additive measures and the Choquet integral. *Annals of Operations Research*, 52:43–65, 1994.

[67] C. Gonzales and P. Perny. GAI networks for utility elicitation. In *Proceedings of the 9th International Conference on the Principles of Knowledge Representation and Reasoning (KR)*, pages 224–234, 2004.

[68] M. Grabisch. The application of fuzzy integrals in multicriteria decision making. *European Journal of Operational Research*, 89:445–456, 1996.

[69] M. Grabisch. k-order additive discrete fuzzy measures and their representation. *Fuzzy Sets and Systems*, 92:167–189, 1997.

[70] M. Grabisch, I. Kojadinovic, and P. Meyer. A review of methods for capacity identification in Choquet integral based multi-attribute utility theory: Applications of the Kappalab R package. *European Journal of Operational Research*, 186(2):766–785, 2008.

[71] M. Grabisch and C. Labreuche. Bi-capacities-I: definition, Möbius transform and interaction. *Fuzzy Sets and Systems*, 151(2):211–236, 2005.

[72] M. Grabisch and C. Labreuche. Bi-capacities-II: the Choquet integral. *Fuzzy Sets and Systems*, 151(2):237–259, 2005.

[73] M. Grabisch and C. Labreuche. Fuzzy measures and integrals in MCDA. In J. Figueira, S. Greco, and M. Ehrgott, editors, *Multiple Criteria Decision Analysis: State of the Art Surveys*, pages 563–604. Springer, Berlin, 2005.

[74] M. Grabisch and C. Labreuche. A decade of application of the Choquet and Sugeno integrals in multi-criteria decision aid. *Annals of Operations Research*, 175(1):247–286, 2010.

[75] M. Grabisch, C. Labreuche, and J.-C. Vansnick. On the extension of pseudo-boolean functions for the aggregation of interacting criteria. *European Journal of Operational Research*, 148(1):28–47, 2003.

[76] M. Grabisch, J.-L. Marichal, and M. Roubens. Equivalent representations of set functions. *Mathematics of Operations Research*, 25(2):157–178, 2000.

[77] M. Grabisch and M. Roubens. Application of the Choquet integral in multicriteria decision making. *Fuzzy Measures and Integrals-Theory and Applications*, pages 348–374, 2000.

[78] S. Greco, M. Kadziński, V. Mousseau, and R. Słowiński. ELECTRE^{GKMS}: Robust ordinal regression for outranking methods. *European Journal of Operational Research*, 214(1):118–135, 2011.

[79] S. Greco, M. Kadziński, and R. Słowiński. Selection of a representative value function in robust multiple criteria sorting. *Computers & Operations Research*, 38:1620–1637, 2011.

[80] S. Greco, B. Matarazzo, and S. Giove. The Choquet integral with respect to a level dependent capacity. *Fuzzy Sets and Systems*, 175:1–35, 2011.

[81] S. Greco, B. Matarazzo, and R. Słowiński. The use of rough sets and fuzzy sets in MCDM. In T. Gal, T. Stewart, and T. Hanne, editors, *Advances in Multiple Criteria Decision Making*, chapter 14, pages 14.1–14.59. Kluwer Academic, Dordrecht, 1999.

[82] S. Greco, B. Matarazzo, and R. Słowiński. Rough sets theory for multicriteria decision analysis. *European Journal of Operational Research*, 129(1):1–47, 2001.

[83] S. Greco, B. Matarazzo, and R. Słowiński. Rough sets methodology for sorting problems in presence of multiple attributes and criteria. *European Journal of Operational Research*, 138(2):247–259, 2002.

[84] S. Greco, B. Matarazzo, and R. Słowiński. Axiomatic characterization of a general utility function and its particular cases in terms of conjoint measurement and rough-set decision rules. *European Journal of Operational Research*, 158:271–292, 2004.

[85] S. Greco, B. Matarazzo, and R. Słowiński. Dominance-based rough set approach to decision under uncertainty and time preference. *Annals of Operations Research*, 176(1):41–75, 2010.

[86] S. Greco, V. Mousseau, and R. Słowiński. Ordinal regression revisited: multiple criteria ranking using a set of additive value functions. *European Journal of Operational Research*, 191(2):416–436, 2008.

[87] S. Greco, V. Mousseau, and R. Słowiński. Multiple criteria sorting with a set of additive value functions. *European Journal of Operational Research*, 207(3):1455–1470, 2010.

[88] S. Greco, V. Mousseau, and R. Słowiński. Robust ordinal regression for value functions handling interacting criteria. *European Journal of Operational Research*, 239(3):711–730, 2014.

[89] S. Greco and F. Rindone. Bipolar fuzzy integrals. *Fuzzy Sets and Systems*, 220(1):21–33, 2013.

[90] S. Greco and F. Rindone. Robust integrals. *Fuzzy Sets and Systems*, 232:18–38, 2013.

[91] G. W. Greenwood, X. S. Hu, and J. G. D'Ambrosio. Fitness functions for multiple objective optimization problems: combining preferences with Pareto rankings. In *Foundations of Genetic Algorithms*, pages 437–455. Morgan Kaufmann, 1997.

[92] A. Ishizaka, D. Balkenborg, and T. Kaplan. Does AHP help us make a choice? An experimental evaluation. *Journal of the Operational Research Society*, 62(10):1801–1812, 2011.

[93] A. Ishizaka and A. Labib. Analystic Hierarchy Process and Expert Choice: benefits and limitations. *OR Insight*, 22(4):201–220, 2009.

[94] A. Ishizaka and M. Lusti. How to derive priorities in AHP: a comparative study. *Central European Journal of Operations Research*, 14(4):387–400, 2006.

[95] A. Ishizaka and P. Nemery. *Multi-criteria decision analysis: methods and software*. John Wiley & Sons, 2013.

[96] A. Ishizaka and N.H. Nguyen. Calibrated fuzzy AHP for current bank account selection. *Expert Systems with Applications*, 40(9):3775–3783, 2013.

[97] E. Jacquet-Lagrèze and Y. Siskos. Assessing a set of additive utility functions for multicriteria decision-making, the UTA method. *European Journal of Operational Research*, 10(2):151–164, 1982.

[98] E. Jacquet-Lagrèze and Y. Siskos. Preference disaggregation: 20 years of MCDA experience. *European Journal of Operational Research*, 130(2):233–245, 2001.

[99] A. Lopez Jaimes, A. Arias Montano, and C. A. Coello Coello. Preference incorporation to solve many-ojective airfoil design problems. In *Congress on Evolutionary Computation*, pages 1605–1612. IEEE, 2011.

[100] A. Jaszkiewicz. Interactive multiobjective optimization with the Pareto memetic algorithm. *Foundations of Computing and Decision Sciences*, 32(1):15–32, 2007.

[101] M. Kadziński, S. Greco, and R. Słowiński. Selection of a representative value function for robust ordinal regression in group decision making. *Group Decision and Negotiation*, pages 1–34, 2011.

[102] M. Kadziński, S. Greco, and R. Słowiński. Selection of a representative value function in robust multiple criteria ranking and choice. *European Journal of Operational Research*, 217(3):541–553, 2011.

[103] M. Kadziński, S. Greco, and R. Słowiński. Extreme ranking analysis in robust ordinal regression. *Omega*, 40(4):488–501, 2012.

[104] M. Kadziński, S. Greco, and R. Słowiński. Selection of a representative set of parameters for robust ordinal regression outranking methods. *Computers & Operations Research*, 2012.

[105] M. Kadziński and T. Tervonen. Robust multi-criteria ranking with additive value models and holistic pair-wise preference statements. *European Journal of Operational Research*, 228(1):169–180, 2013.

[106] M. Kadziński and T. Tervonen. Stochastic ordinal regression for multiple criteria sorting problems. *Decision Support Systems*, 55(11):55–66, 2013.

[107] I. Karahan and M. Köksalan. A territory defining multiobjective evolutionary algorithms and preference incorporation. *IEEE Transactions on Evolutionary Computation*, 14(4):636–664, 2010.

[108] R.L. Keeney. Analysis of preference dependencies among objectives. *Operations Research*, 29:1105–1120, 1981.

[109] R.L. Keeney and H. Raiffa. *Decisions with multiple objectives: Preferences and value tradeoffs*. J. Wiley, New York, 1993.

[110] G.A. Kiker, T.S. Bridges, A. Varghese, T.P. Seager, and I. Linkov. Application of multicriteria decision analysis in environmental decision making. *Integrated environmental assessment and management*, 1(2):95–108, 2005.

[111] J. Knowles. A summary-attainment-surface plotting method for visualizing the performance of stochastic multiobjective optimizers. In *Intelligent Systems Design and Applications, 2005. ISDA'05. Proceedings. 5th International Conference on*, pages 552–557. IEEE, 2005.

[112] P. Korhonen, J. Wallenius, and S. Zionts. Solving the discrete multiple criteria problem using convex cones. *Management Science*, 30:1336–1345, 1984.

[113] R. Lahdelma, J. Hokkanen, and P. Salminen. SMAA - stochastic multiobjective acceptability analysis. *European Journal of Operational Research*, 106(1):137–143, 1998.

[114] R. Lahdelma, K. Miettinen, and P. Salminen. Ordinal criteria in stochastic multicriteria acceptability analysis (SMAA). *European Journal of Operational Research*, 147(1):117–127, 2003.

[115] R. Lahdelma and P. Salminen. SMAA-2: Stochastic multicriteria acceptability analysis for group decision making. *Operations Research*, 49(3):444–454, 2001.

[116] P. Leskinen, J. Viitanen, A. Kangas, and J. Kangas. Alternatives to incorporate uncertainty and risk attitude in multicriteria evaluation of forest plans. *Forest Science*, 52(3):304–312, 2006.

[117] I. Linkov and E. Moberg. *Multi-criteria decision analysis: environmental applications and case studies*. CRC Press, 2011.

[118] J. Malczewski. *GIS and multicriteria decision analysis*. John Wiley & Sons, 1999.

[119] J.L. Marichal and M. Roubens. Determination of weights of interacting criteria from a reference set. *European Journal of Operational Research*, 124(3):641–650, 2000.

[120] G.A. Mendoza and H. Martins. Multi-criteria decision analysis in natural resource management: a critical review of methods and new modelling paradigms. *Forest ecology and management*, 230(1):1–22, 2006.

[121] R. Mesiar, A. Mesiarová-Zemánková, and K. Ahmad. Level-dependent Sugeno integral. *IEEE Transactions on Fuzzy Systems*, 17(1):167–172, 2009.

[122] G.A. Miller. The magical number seven, plus or minus two: some limits on our capacity for processing information. *Psychological review*, 63(2):81, 1956.

[123] I. Millet. The effectiveness of alternative preference elicitation methods in the Analytic Hierarchy Process. *Journal of Multi-Criteria Decision Analysis*, 6(1):41–51, 1997.

[124] F. Modave and M. Grabisch. Preference representation by the Choquet integral: The commensurability hypothesis. *Proceedings of the Seventh International Conference IPMU Paris, July 6–10, 1998*, pages 164–171, 1998.

[125] V. Mousseau, J. Figueira, L. Dias, C. Gomes da Silva, and J. Climaco. Resolving inconsistencies among constraints on the parameters of an MCDA model. *European Journal of Operational Research*, 147(1):72–93, 2003.

[126] V. Mousseau and R. Słowiński. Inferring an ELECTRE TRI model from assignment examples. *Journal of Global Optimization*, 12(2):157–174, 1998.

[127] V. Mousseau, R. Słowiński, and P. Zielniewicz. A user-oriented implementation of the ELECTRE-TRI method integrating preference elicitation support. *Computers & Operations Research*, 27(7-8):757–777, 2000.

[128] S. Murofushi and T. Soneda. Techniques for reading fuzzy measures (III): interaction index. *9th Fuzzy Systems Symposium, Sapporo, Japan*, pages 693–696, 1993.

[129] T. Murofushi, M. Sugeno, and K. Fujimoto. Separated hierarchical decomposition of the Choquet integral. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 05(05):563–585, 1997.

[130] Y. Narukawa and T. Murofushi. The n-step Choquet integral on finite spaces. In *Proc. of the 9th Int. Conf. on Information Proc. and Management of Uncertainty in Knowledge-Based Systems, Annecy, France*, pages 539–543, 2002.

[131] J.A Nelder and R. Mead. A simplex method for function minimization. *The Computer Journal*, 7(4):308–313, 1965.

[132] Comptroller of the Currency Administrator of National Banks. Bank supervision process. Available at: http://www.occ.gov/publications/publications-by-type/comptrollers-handbook/banksupervisionprocess.html, 2007.

[133] D. Pekelman and S.K. Sen. Mathematical programming models for the determination of attribute weights. *Management Science*, 20(8):1217–1229, 1974.

[134] J. Pérez, J.L. Jimeno, and E. Mokotoff. Another potential shortcoming of AHP. *Top*, 14(1):99–111, 2006.

[135] S. Phelps and M. Köksalan. An interactive evolutionary metaheuristic for multiobjective combinatorial optimization. *Management Science*, 49(12):1726–1738, 2003.

[136] M. Pirlot, H. Schmitz, and P. Meyer. An empirical comparison of the expressiveness of the additive value function and the Choquet integral models for representing rankings. In *Proceedings of the 25th Mini-EURO conference on uncertainty and robustness in planning and decision making, URPDM, Coimbra, Portugal*. 2010.

[137] L. Rachmawati and D. Srinivasan. Preference incorporation in multi-objective evolutionary algorithms: A survey. In *Congress on Evolutionary Computation*, pages 3385–3391. IEEE, 2006.

[138] F.S. Roberts. Measurement theory with applications to decision-making, utility and the social sciences. In *Encyclopedia of mathematics and its applications (Vol. 7)*. Reading, MA: Addison-Wesley, 1979.

[139] E. Rokou and K. Kirytopoulos. A calibrated group decision process. *Group Decision and Negotiation*, 23(6):1369–1384, 2014.

[140] G.C. Rota. On the foundations of combinatorial theory I. Theory of Möbius functions. *Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 2:340–368, 1964.

[141] B. Roy. *Multicriteria Methodology for Decision Aiding.* Springer-Verlag US, 1996.

[142] B. Roy. Paradigms and challenges. In J. Figueira, S. Greco, and M. Ehrgott, editors, *Multiple Criteria Decision Analysis: State of the Art Surveys*, pages 3–24. Springer, Berlin, 2005.

[143] B. Roy and D. Bouyssou. *Aide multicritère à la décision: méthodes et cas.* Economica, 1993.

[144] T. Saaty. A scaling method for priorities in hierarchical structures. *Journal of Mathematical Psychology*, 15(3):234–281, 1977.

[145] T. Saaty. *The Analytic Hierarchy Process.* New York, McGraw-Hill, 1980.

[146] T. Saaty. How to make a decision: the Analytic Hierarchy Process. *European Journal of Operational Research*, 48(1):9–26, 1990.

[147] T. Saaty. *Decision Making with Dependence and Feedback: The Analytic Network Process.* RWS Publications, Pittsburgh, 1996.

[148] T.L. Saaty. The Analytic Hierarchy and Analytic Network Processes for the measurement of intangible criteria and for decision-making. In J. Figueira, S. Greco, and M. Ehrgott, editors, *Multiple Criteria Decision Analysis: State of the Art Surveys*, pages 345–382. Springer, Berlin, 2005.

[149] T.L. Saaty and M.S. Ozdemir. Why the magic number seven plus or minus two. *Mathematical and Computer Modelling*, 38(3):233–244, 2003.

[150] R. Sahajwala and P. Van den Bergh. Supervisory risk assessment and early warning systems. Technical Report 4, *Bank of International Settlements*, Basel, Switzerland, December 2000.

[151] L. B. Said, S. Bechikh, and K. Ghedira. The r-dominance: A new dominance relation for interactive evolutionary multicriteria decision making. *IEEE Transactions on Evolutionary Computation*, 14(5):801–818, 2010.

[152] A. Salo and R.P. Hämäläinen. On the measurement of preferences in the Analytic Hierarchy Process. *Journal of Multi-Criteria Decision Analysis*, 6(6):309–319, 1997.

[153] G. Shafer. *A Mathematical Theory of Evidence.* Princeton University Press, 1976.

[154] L.S. Shapley. A value for n-person games. In Tucker A. W. Kuhn, H. W., editor, *Contributions to the Theory of Games II*, pages 307–317. Princeton University Press, Princeton, 1953.

[155] R. Słowiński, S. Greco, and B. Matarazzo. Rough set analysis of preference-ordered data. In J.J Alpigini, J.F. Peters, A. Skowron, and N. Zhong, editors, *Rough Sets and Current Trends in Computing*, volume 2475 of *Lecture Notes in Artificial Intelligence*, pages 44–59. Springer, Berlin, 2002.

[156] R. Słowiński, S. Greco, and B. Matarazzo. Rough-Set-Based Decision Support. In E.K. Burke and G. Kendall, editors, *Search Methodologies: Introductory Tutorials in Optimization and Decision Support Techniques, 2nd edition*, pages 557–609. Springer, New York, 2014.

[157] R. Słowiński, J. Stefanowski, S. Greco, and B. Matarazzo. Rough set based processing of inconsistent information in decision analysis. *Control and Cybernetics*, 29:379–404, 2000.

[158] R.L Smith. Efficient Monte Carlo procedures for generating points uniformly distributed over bounded regions. *Operations Research*, 32:1296–1308, 1984.

[159] J. Spronk, R.E. Steuer, and C. Zopounidis. Multicriteria decision aid/analysis in finance. In J. Figueira, S. Greco, and M. Ehrgott, editors, *Multiple criteria decision analysis: State of the art surveys*, pages 799–848. Springer, Berlin, 2005.

[160] V. Srinivasan and A.D. Shocker. Estimating the weights for multiple attributes in a composite criterion using pairwise judgments. *Psychometrika*, 38(4):473–493, 1973.

[161] T. Stewart. Dealing with Uncertainties in MCDA. In J. Figueira, S. Greco, and M. Ehrgott, editors, *Multiple Criteria Decision Analysis: State of the Art Surveys*, pages 445–460. Springer, Berlin, 2005.

[162] M. Sugeno. *Theory of fuzzy integrals and its applications.* Ph.D. Thesis, Tokyo Institute of Technology, 1974.

[163] M. Sugeno, K. Fujimoto, and T. Murofushi. A hierarchical decomposition of Choquet integral model. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 03(01):1–15, 1995.

[164] T. Tervonen and J. Figueira. A survey on stochastic multicriteria acceptability analysis methods. *Journal of Multi-Criteria Decision Analysis*, 15(1-2):1–14, 2008.

[165] T. Tervonen, J.R. Figueira, R. Lahdelma, D.A. Juscelino, and P. Salminen. A stochastic method for robustness analysis in sorting problems. *European Journal of Operational Research*, 192(1):236–242, 2009.

[166] T. Tervonen, J.R. Figueira, R. Lahdelma, and P. Salminen. SMAA-III: A simulation-based approach for sensitivity analysis of ELECTRE III. In *Real-Time and Deliberative Decision Making*, pages 241–253. Springer, 2009.

[167] T. Tervonen and R. Lahdelma. Implementing stochastic multicriteria acceptability analysis. *European Journal of Operational Research*, 178(2):500 – 513, 2007.

[168] T. Tervonen, G. Van Valkenhoef, N. Basturk, and D. Postmus. Efficient weight generation for simulation based multiple criteria decision analysis. EWG-MCDA, Tarragona, 12-14 April, 2012.

[169] L. Thiele, K. Miettinen, P. J. Korhonen, and J. Molina. A preference-based interactive evolutionary algorithm for multiobjective optimization. *Evolutionary Computation Journal*, 17(3):411–436, 2009.

[170] D. S. Todd and P. Sen. Directed multiple objective search of design spaces using genetic algorithms and neural networks. In W. et al. Banzhaf, editor, *Genetic and Evolutionary Computation Conference*, pages 1738–1743. Morgan Kaufmann, San Francisco, California, 1999.

[171] H. van Greuning and S. Brajovic Bratanovic. *Analyzing Banking Risk - A Framework for Assessing Corporate Governance and Risk Management.* The World Bank, Washington, D.C., 3rd edition, 2009.

[172] G. Van Valkenhoef, T. Tervonen, and D. Postmus. Notes on "Hit-And-Run enables efficient weight generation for simulation-based multiple criteria decision analysis". *European Journal of Operational Research*, 239(3):865–867, 2014.

[173] T. Wagner and H. Trautmann. Integration of preferences in hypervolume-based multiobjective evolutionary algorithms by means of desirability functions. *IEEE Transactions on Evolutionary Computation*, 14(5):688–701, 2010.

[174] P.P. Wakker. *Additive representations of preferences: A new foundation of decision analysis*, volume 4. Springer, 1989.

[175] J.-J. Wang, Y.-Y. Jing, C.-F. Zhang, and J.-H. Zhao. Review on multi-criteria decision analysis aid in sustainable energy decision-making. *Renewable and Sustainable Energy Reviews*, 13(9):2263–2278, 2009.

[176] Y.-M. Wang, K.-S. Chin, and Y. Luo. Aggregation of direct and indirect judgments in pairwise comparison matrices with a re-examination of the criticisms by Bana e Costa and Vansnick. *Information Sciences*, 179(3):329–337, 2009.

[177] J.-Z. Wu and Q. Zhang. 2-order additive fuzzy measure identification method based on diamond pairwise comparison and maximum entropy principle. *Fuzzy Optimization and Decision Making*, 9(4):435–453, 2010.

[178] W. Yu. Aide multicritère à la décision dans le cadre de la problématique du tri: méthodes et applications. Ph.D. thesis. *LAMSADE*, Université Paris Dauphine, Paris, 1992.

[179] C. Zopounidis and M. Doumpos. A multicriteria decision aid methodology for sorting decision problems: the case of financial distress. *Computational Economics*, 14:197–218, 1999.

[180] C. Zopounidis and M. Doumpos. Multicriteria classification and sorting methods: A literature review. *European Journal of Operational Research*, 138:229–246, 2002.