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Solutions of minimal energy for elliptic problems

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Contents

Introduction

This PhD thesis deals with the existence of least energy solutions and least energy nodal ones for nonlinear elliptic problems, characterized by a nonlinearity with some subcritical growth condition.

Nonlinearity elliptic equations are a particular type of partial differential equations written as

$$-Lu = f(\mathbf{x}, u)$$

where L is an elliptic operator and f is a Carathéodory function, i.e. f is measurable in the first variable and continuous in the second one.

We remeber that a second order linear differential operetor L on $u : \Omega \subseteq \mathbb{R}^N \to \mathbb{R}$ (with $N \geq 2$) of the form

$$Lu = -\sum_{i,j=1}^{N} a_{ij}(\mathbf{x}) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(\mathbf{x}) \frac{\partial u}{\partial x_i} + a_0(\mathbf{x}) u$$

where $A(\mathbf{x}) = (a_{ij}(\mathbf{x}))$ is a square matrix of order N, $b(\mathbf{x}) = (b_i(\mathbf{x}))$ is a vector field in $\mathbb{R}^N$ and $a_0 = a_0(\mathbf{x})$ is a real function, is said to be elliptic in Ω if A satisfies the ellipticity condition in Ω, namely if the following condition holds

$$\sum_{i,j=1}^{N} a_{ij}(\mathbf{x})\xi_i\xi_j > \lambda|\xi_i|^2$$

for some constant $\lambda > 0$ (called ellipticity constant) and for all $\mathbf{x} \in \Omega$ and $\xi \in \mathbb{R}^N \backslash \{\mathbf{0}\}$.

Moreover, L is said to be in divergence form if it may be written as

$$Lu = -div(\mathbf{A}(\mathbf{x})\nabla u) + div(\mathbf{b}(\mathbf{x})u) + a_0(\mathbf{x})u$$

which emphasizes the particular structure of the higher order.

Elliptic equations can be studied with different methods and techniques. One of the most elegant and successful method is the variational approach. Probably, this method was originated from the problem of the brachistochrone posed in 1696 and the major contributions were given by Euler (who published the first monograph on the Calculus of Variations) and Lagrange (who introduced formalisms and techniques in use still today).

For the variational approach, the first order terms of elliptic operator have to be neglected and, therefore, L always is in the following form

$$Lu = -div(\mathbf{A}(\mathbf{x})\nabla u) + a_0(\mathbf{x})u$$

When $\mathbf{A}$ is the identity matrix I and $a_0(\mathbf{x}) = 0$, the operator L can be written as

$$Lu = -div(I(\mathbf{x})\nabla u) = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial {x_i}^2} =: -\Delta u$$

and Δ is called Laplace operator or Laplacian.

The classical example of elliptic problem is the Dirichlet problem

$$\begin{cases} -\Delta u = 0 & \text{in} \quad \Omega \\ u = f & \text{on} \quad \partial\Omega \end{cases} \qquad (PD)$$

where Ω is a set in the plane or in the space and f is a function on the boundary of Ω. This problem consists in an elliptic differential equation coupled with a boundary condition and, under suitable assumptions, the solution is the global minimun of the functional

$$I(u) = \int_{\Omega} |\nabla u|^2 dx$$

among all u belonging to a conveniet function space.

This point of view was introduced by Riemann in 1851 and was said Dirichlet Principle.

The fondamental idea is the interpretation of a differential problem as

$$I^{'}(u) = 0 \qquad (1)$$

where I is a functional associated to the problem and $I^{'}$ is its differential in a sense to be made precise (in the most cases it is its Gâteaux differential). The equation (1) is said

Euler (or Euler-Lagrange) equation and the advantage of the Dirichlet principle is that many times finding critical point of I is easier than to work on the differential problem. This method is said Dirichlet method of the Calculus of Variation.

Furthermore, in a lot of applications the functional I has a very important physical meaning: it often represents an energy of some sort and therefore finding a minimum point of I means finding the solution of minimal energy, that has a particular relevance in concrete problems. The physical interpretation of I is so important that the functionals I associated with the problems are always called Energy Functional. But, of course, not all problems can be written in the form (1). When it is possible, one says that the problem has a variational structure.

In this PhD thesis, we apply the variational method to the following problem

$$\begin{cases} -\Delta u = f(x,u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{P}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ and f is a Carath$\acute{e}$odory function with suitable growth condition, in order to prove the existence of least energy solutions and least energy nodal ones.

To this aim, the thesis is divided into three chapters. The first chapter is devoted to basic notions and results, which will be needed to prove our main results. In fact, in many problems of variational calculus, it is not sufficient to deal with the classical solutions of differential equations, but it is necessary to introduce the notion of weak solutions and to work in the so called Sobolev spaces. In the second chapter, we prove a general existence result of least energy solutions and least energy nodal ones for the problem (P). The last chapter deals with some previous results related to special cases of f. Finally, we propose some open questions concerning the global minima of the restriction on the Nehari manifold of the energy functional associated with (P), when the nonlinearity is of the type

$$f(x,u) = \lambda|u|^{s-2}u - \mu|u|^{r-2}u$$

with $s, r \in (1,2)$ and $\lambda, \mu > 0$.

At last, we want to point out that our Bibliography includes some references, which were used to describe the introductory topics in the first chapter and are not explicitly mentioned in the thesis.

Chapter 1

Preliminaries

In this section we introduce the basic notions and results, which we will use to prove our main results in the second chapter.

1.1 Sobolev spaces

Sobolev spaces are vector spaces whose elements are fuctions defined on domains in N-dimensional Euclidean space $\mathbb{R}^N$ and whose partial derivatives satisfy integrability conditions. Their name is due to the Russian mathematician Sergei Lvovich Sobolev.

1.1.1 The spaces $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$

In order to describe these spaces, we have to put forward an important definition.

Definition 1.1

Let Ω be an open subset of $\mathbb{R}^N$. Let $C_c^\infty(\Omega)$ denote the space of infinitely differentiable functions $\varphi : \Omega \to \mathbb{R}$, with compact support in Ω. A function φ belonging to $C_c^\infty(\Omega)$ is called test function.

Definition 1.2

Let Ω be an open subset of $\mathbb{R}^N$ and $p \geq 1$. The set

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) : \exists \quad g_1, g_2, \cdots, g_N \in L^p(\Omega) \ \ s.t. \ \ \int_\Omega u \frac{\partial \varphi}{\partial x_i} = -\int_\Omega g_i \varphi \right.$$
$$\left. \forall \ \ \varphi \in C_c^\infty(\Omega) \ \ , \ \ \forall \ \ i = 1, 2, \cdots, N \right\}$$

is called Sobolev space. When $u \in W^{1,p}(\Omega)$, the function g_i is called weak derivative (or derivative in the sense of distribution) of u with respect to the i-th variable x_i, and it is usually denoted by $\partial_i u$. Moreover, the vector $\nabla u := (\partial_i u, ..., \partial_N u)$ is called weak gradient of u. Clearly, $|\nabla u| \in L^p(\Omega)$ if $u \in W^{1,p}(\Omega)$.

We can endow this space with the following norm

$$\|u\|_{W^{1,p}} = (\|u\|_{L^p}^p + \|\nabla u\|_{L^p}^p)^{\frac{1}{p}} = \left(\int_\Omega |u|^p dx + \int_\Omega |\nabla u|^p dx \right)^{\frac{1}{p}}$$

if $1 \leq p < \infty$, and

$$\|u\|_{W^{1,\infty}} = \max\{\|u\|_{L^\infty}, \|\nabla u\|_{L^\infty}\}$$

if $p = \infty$.

Equipped with this norm, the space $W^{1,p}(\Omega)$ is a Banach space and, under suitable conditions, it is separable and reflexive.

In fact, the following theorem holds.

Theorem 1.1

Let Ω be an open set in $\mathbb{R}^N$. Then

(1) *$W^{1,p}(\Omega)$ is a Banach space if $1 \leq p \leq \infty$.*

(2) *$W^{1,p}(\Omega)$ is separeble if $1 \leq p < \infty$.*

(3) *$W^{1,p}(\Omega)$ is reflexive if $1 < p < \infty$.*

Remark 1.1 Clearly, $W^{0,p}(\Omega) = L^p(\Omega)$.

We want to give an easy characterization of functions belong to $W^{1,p}(\Omega)$. To this aim, we need to prefece the following notions.

Definition 1.3

Let $\Omega \subset \mathbb{R}^N$ be an open set. We say that an open set ω in $\mathbb{R}^N$ is strongly included in Ω and we write $\omega \subset\subset \Omega$ if $\bar{\omega} \subset \Omega$ and $\bar{w}$ is compact.

Notation. Given $x \in \mathbb{R}^N$, write

$$x = (x^{'}, x_N)$$

with $x^{'} \in \mathbb{R}^{N-1}$ and set

$$|x^{'}| = \left(\sum_{i=1}^{N-1} {x_i}^2 \right)^{\frac{1}{2}}$$

We define

$$\mathbb{R}^N_+ = \{x = (x^{'}, x_N); x_N > 0\}$$

$$Q = \{x = (x^{'}, x_N); |x^{'}| < 1 \ \ and \ \ |x_N| < 1\}$$

$$Q_+ = Q \cap \mathbb{R}^N_+$$

$$Q_0 = \{x = (x^{'}, 0); |x^{'}| < 1\}$$

Definition 1.4

An open set Ω is of class C^1 if for every $x \in \partial\Omega$ there exist a neighborhood U of $x \in \mathbb{R}^N$ and a bijective map $H : Q \to U$ such that $H \in C^1(\bar{Q})$, $H^{-1} \in C^1(\bar{U})$, $H(Q_+) = U \cap Q$

and $H(Q_0) = U \cap \partial\Omega$.

The map H is called a local chart.

Now. it is possible to prove the characterization of functions in $W^{1,p}(\Omega)$.

Proposition 1.1

Let $u \in L^p(\Omega)$ with $1 < p \leq \infty$. The following properties are equivalent:

(i) $u \in W^{1,p}(\Omega)$;

(ii) *there exists a constant C such that*

$$\left| \int_\Omega u \frac{\partial \varphi}{\partial x_i} \right| \leq C \|\varphi\|_{L^{p'}(\Omega)}$$

for every $\varphi \in C_c^\infty(\Omega)$ and for every $i = 1, 2, \cdots, N$, where p' is the conjugate exponent of p $\left(i.e\ \frac{1}{p} + \frac{1}{p'} = 1 \right)$;

(iii) *there exists a constant C such that for every $\omega \subset\subset \Omega$ and all $h \in \mathbb{R}^N$ with $|h| < dist(\omega, \partial\Omega)$, we have*

$$\|u((\cdot) + h) - u\|_{L^p(\omega)} \leq C|h|$$

Furthermore, we can take $C = \|\nabla u\|_{L^p(\Omega)}$ in (ii) and (iii).

We have

$$\|u((\cdot) + h) - u\|_{L^p(\mathbb{R}^N)} \leq |h| \|\nabla u\|_{L^p(\mathbb{R}^N)}$$

if $\Omega = \mathbb{R}^N$.

Proof.

$(i) \quad \Rightarrow \quad (ii)$: Obvious

$(ii) \quad \Rightarrow \quad (i)$

The linear functional $\varphi \in C_c^\infty(\Omega) \to \int_\Omega u\varphi'$ is defined on a dense subspace of $L^{p'}$ (since $p' < \infty$) and it is continuous for the $L^{p'}$ norm. Therefore, it extends to a bounded linear functional F defined on all $L^{p'}$ (applying the Hahn-Banach theorem). By the Riesz representation theorems, there exists $g \in L^p$ such that

$$(F, \varphi) = \int_\Omega g\varphi$$

for every $\varphi \in L^{p'}$. In particular,

$$\int_\Omega u\varphi' = \int_\Omega g\varphi$$

for every $\varphi \in C_c^\infty$ and thus $u \in W^{1,p}$.

$(i) \quad \Rightarrow \quad (iii)$

Assume first that $u \in C_c^\infty(\mathbb{R}^N)$. Let $h \in \mathbb{R}^N$ and set $v_x(t) = u(x + th)$ with $t \in \mathbb{R}$ and $x \in \mathbb{R}$.

Then, $v_x^{'}(t) = h \cdot \nabla u(x + th)$ and thus

$$u(x+h) - u(x) = v_x(1) - v_x(0) = \int_0^1 v_x^{'}(t)dt = \int_0^1 h \cdot \nabla u(x+th)dt$$

Then, it follows that for $1 \leq p < \infty$,

$$|u((\cdot)+h) - u(x)|^p \geq |h|^p \int_0^1 |h \cdot \nabla u(x+th)|^p dt$$

and

$$\begin{aligned}\int_\omega |u((\cdot)+h) - u(x)|^p dx \leq \ & |h|^p \int_\omega dx \int_0^1 |\nabla u(x+th)|^p dt = \\ & = |h|^p \int_0^1 dt \int_\omega |\nabla u(x+th)|^p dx = \\ & = |h|^p \int_0^1 dt \int_{\omega+th} |\nabla u(y)|^p dy\end{aligned}$$

If $|h| < dist(\omega, \partial\Omega)$, there exists an open set $\omega^{'} \subset\subset \Omega$ such that $\omega + th \subset \omega^{'}$ for every $t \in [0,1]$ and thus

$$\|u((\cdot)+h) - u\|_{L^p(\omega)}^p \leq |h|^p \int_{\omega^{'}} |\nabla u|^p \qquad (*)$$

This concludes the proof of (ii) for $u \in C_0^\infty(\mathbb{R}^N)$ and $1 \leq p < \infty$.

Now assume that $u \in W^{1,p}(\Omega)$ with $1 \leq p < \infty$. In this case, there exists a sequence (u_n) in $C_c^\infty(\mathbb{R}^N)$ such that $u_n \to u$ in $L^p(\Omega)$ and $\nabla u_n \to \nabla u$ in $L^p(\omega)^N$ for every $\omega \subset\subset \Omega$. Applying $(*)$ to (u_n) and passing to the limit, we obtain (iii) for every $u \in W^{1,p}(\Omega)$, $1 \leq p < \infty$. When $p = \infty$, apply the above (for $p < \infty$) and let $p \to \infty$.

$(iii) \quad \Rightarrow \quad (ii)$

Let $\varphi \in C_c^\infty(\Omega)$ and consider an open set ω such that $supp\varphi \subset \omega \subset\subset \Omega$. Let $h \in \mathbb{R}^N$ with $|h| < dist(\omega, \partial\Omega)$. Because of (iii) we have

$$\left| \int_\Omega (u((\cdot) + h) - u)\varphi \right| \leq C|h| \|\varphi\|_{L^{p'}(\Omega)}$$

On the other hand, since

$$\int_\Omega (u(x+h) - u(x))\varphi(x)dx = \int_\Omega u(y)(\varphi(y-h) - \varphi(y))dy$$

it follows that

$$\int_\Omega u(y)\frac{\varphi(y-h) - \varphi(y)}{|h|}dy \leq C\|\varphi\|_{L^{p'}}$$

Choosing $h = te_i$, $t \in \mathbb{R}$, and passing to the limit as $t \to 0$, we obtain (ii). $\square$

Remark 1.2 When $p = 1$ the following implications remain true:

$$(i) \quad \Rightarrow \quad (ii) \quad \Leftrightarrow \quad (iii)$$

The functions that satisfy (ii) (or (iii)) with $p = 1$ are called functions of bounded variation (in the language of distributions a function of bounded variation is an L^1 function such that all its first derivatives, in the sense of distributions, are bounded measures).

Remark 1.3 If $\Omega \subset \mathbb{R}$, the space $W^{1,p}(\Omega)$ is the space of all function u such that there exists an absolutely countinuos function $f : \Omega \to \mathbb{R}$ such that $f(t) = u(t)$ almost everywhere in Ω and f' belongs to $L^p(\Omega)$. In particular, in $W^{1,1}(\Omega)$, there are only the absolutely continuous function.

We now pass to define a particular subspace of $W^{1,p}(\Omega)$, the space $W_0^{1,p}(\Omega)$, which plays a very interesting role in studying the Dirichlet problem associated to elliptic equations.

Definition 1.5

Let $1 \leq p < \infty$; $W_0^{1,p}(\Omega)$ denotes the closure of $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$, that is

$$W_0^{1,p}(\Omega) = \overline{(C_c^\infty(\Omega))}_{W^{1,p}(\Omega)}$$

The space $W_0^{1,p}(\Omega)$ can be equipped with the $W^{1,p}(\Omega)$ norm and it is a separeble Banach space. Moreover, if $1 < p < \infty$, the space $W_0^{1,p}(\Omega)$ is reflexive.

Remark 1.4 Since $C_c^\infty(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N)$, we have

$$W_0^{1,p}(\mathbb{R}^N) = W^{1,p}(\mathbb{R}^N)$$

By contrast, if $\Omega \subseteq \mathbb{R}^N$, then in general, $W_0^{1,p}(\Omega) \neq W^{1,p}(\Omega)$.

However, if $\mathbb{R}^N \backslash \Omega$ is sufficiently thin and $p < N$, then $W_0^{1,p}(\mathbb{R}^N) = W^{1,p}(\mathbb{R}^N)$.

The functions in $W_0^{1,p}(\Omega)$ are those of $W^{1,p}(\Omega)$ that vanish on $\partial\Omega$ in some sense. It is delicate to make this precise, since a function $u \in W^{1,p}(\Omega)$ is defined only almost everywhere, the measure of $\partial\Omega$ is zero and u need not have a continuous representative. But, we will prove a fundamental characterization, which shows that the continuous functions of $W_0^{1,p}(\Omega)$ are functions that are really zero on $\partial\Omega$. In order to prove that characrerization, we give first the following lemma.

Lemma 1.1

Let $u \in W^{1,p}(\Omega)$ with $1 \leq p < \infty$ and assume that supp(u) is a compact subset of Ω. Then, $u \in W_0^{1,p}(\Omega)$.

Theorem 1.2

Let Ω be of class C^1 and $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ with $1 \leq p < \infty$.

Then, the following properties are equivalent:

(i) $u = 0$ *on* $\partial\Omega$.

(ii) $u \in W_0^{1,p}(\Omega)$.

Proof.

$(i) \;\; \Rightarrow \;\; (ii)$

Suppose first that $supp(u)$ is bounded. Fix a function $G \in C^1(\mathbb{R})$ such that $|G(t)| \leq |t|$ for every $t \in \mathbb{R}$ and $G(t) = \begin{cases} 0 & \text{if } |t| \leq 1 \\ 1 & \text{if } |t| \geq 2 \end{cases}$

Then, $u_n = \left(\frac{1}{n}\right) G(u_n)$ belongs to $W^{1,p}(\Omega)$ and $u_n \to u$ in $W^{1,p}(\Omega)$. On the other hand,

$$supp(u_n) \subset \left\{ x \in \Omega; |u(x)| \geq \frac{1}{n} \right\}$$

and thus $supp(u_n)$ is a compact set contained in Ω. From Lemma (1.1), $u_n \in W_0^{1,p}(\Omega)$ and it follows that $u \in W_0^{1,p}(\Omega)$.

In the general case in which $supp(u)$ is not bounded, consider the sequence $(\xi_n u)$, where (ξ_n) is a sequence of cut-off functions defined as follows: fixed a function $\xi \in C_c^\infty(\mathbb{R}^N)$, with $0 \leq \xi \leq 1$ and

$$\xi(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| \geq 2 \end{cases}$$

we set $\xi_n(x) = \xi\left(\frac{x}{n}\right)$, with $n = 1, 2, \cdots$

From the above, $\xi_n u \in W_0^{1,p}(\Omega)$ and, since $\xi_n u \to u$ in $W^{1,p}(\Omega)$, we conclude that $u \in W_0^{1,p}(\Omega)$.

$(ii) \;\Rightarrow\; (i)$

Using local chart this is reduced to the following problem. Let $u \in W_0^{1,p}(Q_+) \cap C(\bar{Q}_+)$; prove that $u = 0$ on Q_0.

Let (u_n) be a sequence in $C_c^\infty(Q_+)$ such that $u_n \to u$ in $W^{1,p}(Q_+)$.

For $(x^{'}, x_N) \in Q_+$, we have

$$|u_n(x^{'}, x_N)| \leq \int_0^{x_n} \left| \frac{\partial u_n}{\partial x_n}(x^{'}, t) \right| dt$$

and thus for $0 < \epsilon < 1$,

$$\frac{1}{\epsilon} \int_{|x'|<1} \int_0^\epsilon |u_n(x^{'}, x_N)| dx^{'} dx_N \leq \frac{1}{\epsilon} \int_{|x'|<1} \int_0^\epsilon \left| \frac{\partial u_n}{\partial x_n}(x^{'}, t) \right| dt$$

In the limit, when $n \to \infty$ and $\epsilon > 0$ fixed we obtain

$$\frac{1}{\epsilon} \int_{|x'|<1} \int_0^\epsilon |u(x^{'}, x_N)| dx^{'} dx_N \leq \frac{1}{\epsilon} \int_{|x'|<1} \int_0^\epsilon \left| \frac{\partial u}{\partial x_n}(x^{'}, t) \right| dt$$

Finally, as $\epsilon \to 0$, we are led to

$$\int_{|x'|<1} |u(x^{'}, 0)| dx^{'} = 0$$

since $u \in C(\bar{Q}_+)$ and $\dfrac{\partial u}{\partial x_N} \in L^1(Q_+)$. Thus $u = 0$ on Q_0. □

Here it is another characterization of $W_0^{1,p}(\Omega)$.

Proposition 1.2

Let Ω be of class C^1 and let $u \in L^p(\Omega)$ with $1 < p < \infty$. The following properties are equivalent:

(i) $u \in W_0^{1,p}(\Omega)$;

(ii) *there exists a constant C such that*

$$\left| \int_\Omega u \frac{\partial \varphi}{\partial x_i} \right| \leq C \|\varphi\|_{L^{p'}(\Omega)}$$

for every $\varphi \in C_c^\infty(\mathbb{R}^N)$ and for every $i = 1, 2, \cdots, N$;

(iii) *the function*

$$\bar{u}(x) = \begin{cases} u(x) & \text{if } \ \mathrm{x} \in \Omega \\ 0 & \text{if } \ \mathrm{x} \in \mathbb{R}^\mathrm{N} \setminus \Omega \end{cases}$$

belongs to $W^{1,p}(\mathbb{R}^N)$, and in this case $\dfrac{\partial \overline{u}}{\partial x_i}$.

An important corollary is represented by Poincaré's inequality.

Corollary 1.1 *[Poincaré's inequality]*

Suppose that $1 \leq p < \infty$ and Ω is a bounded open set. Then, there exists a constant C, depending on Ω and p, such that

$$\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}$$

for every $u \in W_0^{1,p}(\Omega)$.

In particular, the expression $\|\nabla u\|_L^p(\Omega)$ is a norm on $W_0^{1,p}(\Omega)$.

Remark 1.5 Poincaré's inequality remains true if Ω has finite measure and also if Ω has a bounded projection on some axis.

1.1.2 Special case: $p = 2$

The case of $p = 2$ is a particular case for Sobolev space.

If we put $W^{1,2}(\Omega) = H^1(\Omega)$, the space $H^1(\Omega)$ is

$$H^1(\Omega) = \left\{ u \in L^2(\Omega) : \frac{\partial u}{\partial x_i} \in L^2(\Omega), i = 1, 2, \cdots, N \right\}$$

where the derivative $\dfrac{\partial u}{\partial x_i}$ is in the sense of distributions.

It is a Hilbert space, when it is endowed with the scalar product given by

$$(u, v) = \int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega uv dx$$

The corresponding norm is

$$\|u\|_{H^1} = \sqrt{(u, u)} = \left(\int_\Omega |\nabla u|^2 dx + \int_\Omega u^2 dx \right)^{\frac{1}{2}}$$

The space $H_0^1(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$.

A useful extension property of $H_0^1(\Omega)$ is the following.

Proprerty 1.1

Let $u \in H_0^1(\Omega)$ and let Ω' be an open set such that $\Omega \subset \Omega'$. If one exends u to Ω' by setting

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \Omega' \setminus \Omega \end{cases}$$

then $\tilde{u} \in H_0^1(\Omega)$.

Remark 1.6 Also in this case, we can point out that

$$H_0^1(\mathbb{R}^N) = H^1(\mathbb{R}^N)$$

Proprerty 1.2

Let $u \in H^1(\Omega)$. Then

(1) $|u| \in H^1(\Omega)$;

(2) *if we set $u_+(x) = \max\{u(x), 0\}$ and $u_-(x) = -\max\{u(x), 0\}$, the functions u_+ and u_- belong to $H^1(\Omega)$.*

In the case of $H_0^1(\Omega)$, the Poincaré's inequality, expressed by Corollary 1.1, assumes the following form.

Theorem 1.3 *[Poincaré's inequality] Let $\Omega \subseteq \mathbb{R}^N$ be open and bounded. Then, there exists a constant $C > 0$, depending only on Ω, such that*

$$\int_\Omega u^2 dx \leq C \int_\Omega |\nabla u|^2 dx$$

for every $u \in H_0^1(\Omega)$.

As a consequence, the quantity $\left(\int_\Omega |\nabla u|^2 dx \right)^{\frac{1}{2}}$ is a norm on $H_0^1(\Omega)$, equivalent to the standard one.

1.1.3 Embeddings

The embeddings theorems of Sobolev spaces are essential tools in variational analysis, especially in the study of differential and integral operators. The most important embedding results for Sobolev spaces are often collected into a single theorem, called ”Sobolev embedding theorem”. In order to deal with embeddings theorems, we need to put forward two important definitions.

Definition 1.6

Let X and Y two Banach spaces. Then, X is said to be embedded continuously in Y and it is written $X \hookrightarrow Y$ if

(1) $X \subseteq Y$;

(2) the canonical injection $j : X \to Y$ is a continuous linear operator, namely there exists a costant $C > 0$ such that $\|j(u)\|_Y \leq C\|u\|_X$ which one writes $\|u\|_Y \leq C\|u\|_X$, for every $x \in X$.

Definition 1.7

Let X and Y two Banach spaces. Then, X is said to be embedded compactly in Y if X is embedded continuously in Y and the canonical injection $j : X \to Y$ is a compact linear operator, namely for every bounded subset A in X, the set $\overline{j(A)}$ is a compact set in Y.

At the beginning, we want to deal with the case $\Omega = \mathbb{R}^N$.

Theorem 1.4 *[Sobolev, Gagliardo, Nirenberg]*

Let $1 \leq p < N$. Then

$$W^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$$

where p^* is given by $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{N}$ and there exists a costant $C = C(p, N)$ such that

$$\|u\|_{p^*} \leq C \|\nabla u\|_p$$

for every $u \in W^{1,p}(\mathbb{R}^N)$.

Two important corollaries follow.

Corollary 1.2

Let $1 \leq p < N$. Then

$$W^{1,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$$

for every $q \in [p, p^*]$.

Proof.

Given $q \in [p, p^*]$, we write

$$\frac{1}{q} = \frac{\alpha}{p} + \frac{1-\alpha}{p^*}$$

for some $\alpha \in [0,1]$. By Young's inequality, we obtain

$$\|u\|_q \leq \|u\|_p^{\alpha} \|u\|_{p^*}^{1-\alpha} \leq \|u\|_p + \|u\|_{p^*}$$

Using Theorem (1.4), we conclude that

$$\|u\|_q \leq C\|u\|_{W^{1,p}}$$

for every $u \in W^{1,p}(\mathbb{R}^N)$. □

Corollary 1.3 *[the limiting case $p = N$]*

One has

$$W^{1,N}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$$

for every $q \in [N, +\infty)$.

Another fundamental result is the Morrey theorem.

Theorem 1.5 *[Morrey]*

Let $p > N$. Then

$$W^{1,p}(\mathbb{R}^N) \hookrightarrow L^{\infty}(\mathbb{R}^N)$$

Furthermore, for every $u \in W^{1,p}(\mathbb{R}^N)$, *we have*

$$|u(x) - u(y)| \leq C|x - y|^\alpha \|\nabla u\|_p \qquad (*)$$

almost every $x, y \in \mathbb{R}^N$, *where* $\alpha = 1 - \dfrac{N}{p}$ *and* C *is a constant depending only on* p *and* N.

Remark 1.7 Inequality $(*)$ implies the existence of a function $\tilde{u} \in C(\mathbb{R}^N)$ such that $u = \tilde{u}$ almost everywhere on $\mathbb{R}^N$. In other words, every function $u \in W^{1,p}(\mathbb{R}^N)$ with $p > N$ admits a continuous representative.

Now, we suppose that Ω is an open set of $\mathbb{R}^N$.

Corollary 1.4

Let $1 \leq p \leq \infty$. *We have*

$$W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega) \qquad \text{where } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{N} \quad \text{if } \; p < N$$

$$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \qquad \text{for all } \; q \in [p, +\infty[\quad \text{if } \; p = N$$

$$W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega) \qquad \text{if } \; p > N$$

Moreover, if $p > N$ *we have, for every* $u \in W^{1,p}(\Omega)$,

$$|u(x) - u(y)| \leq C\|u\|_{W^{1,p}}(\Omega)|x - y|^\alpha$$

almost every $x, y \in \Omega$, *with* $\alpha = 1 - \dfrac{N}{p}$ *and* C *depends only on* Ω, p *and* N.

In particular,

$$W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$$

The main compact embedding result is the Rellich-Kondrachov theorem.

Theorem 1.6 *[Rellich-Kondrachov]. Suppose that* Ω *is an open bounded set in* $\mathbb{R}^N$ *and of class* C^1. *Then, we have the following compact embeddings:*

$$W^{1,p}(\Omega) \to L^q(\Omega) \qquad for \;\; all \;\; q \in [1, p^*[, \;\; where \;\; \frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}, \;\; if \;\; p < N$$

$$W^{1,p}(\Omega) \to L^q(\Omega) \qquad for \;\; all \;\; q \in [1, +\infty[\qquad\qquad if \;\; p = N$$

$$W^{1,p}(\Omega) \to L^\infty(\Omega) \qquad\qquad if \;\; p > N$$

In particular, $W^{1,p}(\Omega) \to L^p(\Omega)$ *with compact injection for every* p *(and all* N*).*

Finally, we want to give two results, related with the case of $H^1_0(\Omega)$ and $H^1(\mathbb{R}^N)$.

Theorem 1.7

Let $\Omega \subseteq \mathbb{R}^N$ *be an open and bounded subset of* $\mathbb{R}^N$, *with* $N \geq 3$. *Then*

$$H^1_0(\Omega) \hookrightarrow L^q(\Omega)$$

for every $q \in \left[1, \dfrac{2N}{N-2}\right]$.

The embedding is compact if and only if $q \in \left[1, \dfrac{2N}{N-2}\right[$.

Remark 1.8 The number $\dfrac{2N}{N-2}$ is denoted by 2^* and is called the critical Sobolev exponent for the embedding of $H_0^1(\Omega)$ into $L^q(\Omega)$. The term "critical" refers to the fact that the embedding of the preceding theorem fails for $q > 2^*$.

For functions defined on $\Omega = \mathbb{R}^N$, the following theorem holds.

Theorem 1.8

Let $N \geq 3$. Then

$$H^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$$

for every $q \in [2, 2^*]$ and the embedding is never compact.

We point out that the continuity of the above embeddings is expressed explicitly by inequalities of the form

$$|u|_{L^q} \leq C\|u\|_{H_0^1}$$

for every $u \in H_0^1(\Omega)$, where C does not depend on u.

Inequalities of this type are often referred to as Sobolev inequalities.

1.2 Fréchet and G$\hat{a}$teaux differentiability

We present a short review of the main definitions and results concerning the differential calculus for real functionals defined on a Banach space.

Definition 1.8

Let X be a Banach space, we denote by X' its topological dual, namely the space of continuous linear functionals from X to $\mathbb{R}$.

This space is a Banach space endowed with the norm

$$\|A\| = \sup_{\substack{u \in X \\ \|u\|=1}} |A(u)|$$

Definition 1.9

Let X be a Banach space and $U \subseteq X$, a functional I on U is an application $I : U \to \mathbb{R}$.

1.2.1 Fréchet differentiability

Definition 1.10

Let X be a Banach space, U an open subset of X and let $I : U \to \mathbb{R}$ be a functional.

The functional I is Fréchet differentiable at $u \in U$ if there exists $A \in X'$ such that

$$\lim_{\|v\| \to 0} \frac{I(u+v) - I(u) - Av}{\|v\|} = 0$$

Remark 1.9 For a differentiable functional I, we have

$$I(u+v) - I(u) = Av + o(\|v\|)$$

as $\|v\| \to 0$ for some $A \in X^{'}$, that is the increment $I(u+v) - I(u)$ is asymptotically linear in v as $\|v\| \to 0$. This implies that if I is Fréchet differentiable at u, then it is continuous at u.

Proprerty 1.3

Let I be a functional on U Fréchet differentiable at $u \in U$. Then, there exists a unique element $A \in X^{'}$, which satisfies the definition (1.10).

Proof.

Assume A and B two different elements of $X^{'}$ that satisfy the definition (1.10), then plainly

$$\lim_{\|v\| \to 0} \frac{(A-B)v}{\|v\|} = 0$$

so that , if $u \in X^{'}$ and $\|u\| = 1$

$$(A-B)u = \lim_{t \to 0^{+}} \frac{(A-B)tu}{t} = 0$$

which means $A = B$. $\square$

Definition 1.11

Let $U \subseteq X$ be an open set. The unique element of $A \in X^{'}$ is called the Fréchet differential of I at u, it is denoted by $I^{'}(u)$ and we have

$$I(u+v) = I(u) + I^{'}(u)v + o(\|v\|)$$

as $\|v\| \to 0$. Of course, if the functional I is differentiable at every $u \in U$, then I is differentiable on U. Finally, we notice that the map $I^{'} : U \to X$ that sends $u \in U$ to $I^{'}(u) \in X^{'}$ is called the Fréchet derivative of I and is in general a nonlinear map.

A particular case is that of real functionals defined on a Hilbert space H with scalar product $(\cdot, \cdot)$. Indeed, it is well known that $H^{'}$ and H can be identified via the Riesz isomorphism $R : H^{'} \to H$ and the linear functionals on H can be represented by the scalar product in H as follows: given $A \in H^{'}$ there exists a unique $RA \in H$ such that

$$A(u) = (RA, u)$$

for every $u \in H$.

This allows us to give the following definition.

Definition 1.12

Let H be a Hilbert space, $U \subseteq H$ an open set and let $R : H^{'} \to H$ be the Riesz isomorphism. Assume that the functional $I : U \to \mathbb{R}$ is differentiable at u. The element $RI^{'}(u) \in H$ is called the gradient of I at u and is denoted by $\nabla I(u)$; therefore

$$I^{'}(u)v = (\nabla I(u), v)$$

for every $v \in H$.

The following proposition collects the properties of Fréchet differentiable functionals.

Proposition 1.3

Assume that I and J are differentiable at $u \in U \subseteq X$. Then, the following properties hold:

(1) *If a and b are real numbers, $aI + bJ$ is differentiable at u and*

$$(aI + bJ)^{'}(u) = aI^{'}(u) + bJ^{'}(u)$$

(2) *The product IJ is differentiable at u and*

$$(IJ)^{'}(u) = J(u)I^{'}(u) + I(u)J^{'}(u)$$

(3) *If $\gamma : \mathbb{R} \to U$ is differentiable at t_0 and $u = \gamma(t_0)$, then the composition $\eta : \mathbb{R} \to \mathbb{R}$ defined by $\eta(t) = I(\gamma(t))$ is differentiable at t_0 and*

$$\eta^{'}(t_0) = I^{'}(u)\gamma^{'}(t_0)$$

(4) *If $A \subseteq \mathbb{R}$ is an open set, $f : A \to \mathbb{R}$ is differentiable at $I(u) \in A$, then the composition $K(u) = f(I(u))$ is defined in an open neighborhood V of u, it is differentiable at u and*

$$K^{'}(u) = f^{'}(I(u))I^{'}(u)$$

1.2.2 Gâteaux differentiability

We now introduce a second notion of differentiability, that is often simpler to check in concrete cases than Fréchet differentiability.

Definition 1.13

Let X be a Banach space, $U \subseteq X$ an open set and let $I : U \to \mathbb{R}$ be a functional. The functional I is Gâteaux differentiable at $u \in U$ if there exists $A \in X'$ such that, for every $v \in X$,

$$\lim_{t \to 0} \frac{I(u+tv) - I(u)}{t} = Av$$

If I is Gâteaux differentiable at u, the linear functional $A \in X'$ is unique and it is called the Gâteaux differential of I at u. It is denoted by $I'_G(u)$.

Remark 1.10 It is obvious that if I is Fréchet differentiable at u, then it is also Gâteaux differentiable and $I'(u) = I'_G(u)$. In general, it is not true that G*âteaux* differentiability implies Fréchet differentiability.

Proposition 1.4 *Let $U \subseteq X$ be an open set and I a Gâteaux differentiable functional on U. Given $u, v \in U$ such that the segment $[u, v] = \{tu + (1-t)v : t \in [0, 1]\} \subseteq U$. Then, there results*

$$\|I(u) - I(v)\| \leq \sup\{I'_G(w) : w \in [u, v]\}|u - v\|$$

The following proposition establishes the relationship between the two type of differentiability.

Proposition 1.5

Let $U \subseteq X$ be an open set and I a Gâteaux differentiable functional on U. Assume that I'_G is continuous at $u \in U$. Then, I is also Fréchet differentiable at u and $I'_G(u) = I'(u)$.

Proof.

We set

$$R(h) := I(u+h) - I(u) - I'_G(u)h$$

Plainly, R is Gâteaux differentiable in the ball B_ϵ, with $\epsilon > 0$ small enough, and

$$R'_G(h) : k \to I'_G(u+h)k - I'_G(u)k \qquad (*)$$

Applying Proposition 1.4, with $[u, v] = [0, h]$, being $R(0) = 0$, we find

$$\|R(h)\| \leq \sup_{0 \leq t \leq 1} \|R'_G(th)\| \|h\| \qquad (**)$$

From $(*)$, with th instead h, we deduce

$$\|R'_G(th)\| = \|I'_G(u+th) + I'_G(u)\|$$

Substituing in $(**)$, we find

$$\|R(h)\| \leq \sup_{0\leq t\leq 1} \|I_G^{'}(u+th) + I_G^{'}(u)\|\|h\|$$

Since $I_G^{'}$ is continuous

$$\sup_{0\leq t\leq 1} \|I_G^{'}(u+th) + I_G^{'}(u)\| \to 0 \quad as \quad \|h\| \to 0$$

and therefore $R(h) = o(\|h\|)$.

Thus

$$I_G^{'}(u)h = I(u+h) - I(u) - o(\|h\|)$$

namely

$$I_G^{'}(u)h = I^{'}(u)h$$

□

Remark 1.11 The importance of this proposition lies in the fact that it is often technically easier to compute the Gâteaux derivative and then prove that it is continuous, rather than proving directly the Fréchet differentiability.

We conclude with the definitions of critical points.

Definition 1.14

Let X be a Banach space, $U \subseteq X$ an open set and assume that $I : U \to \mathbb{R}$ is Gâteaux differentiable. A critical point of I is a point $u \in U$ such that

$$I'(u) = 0$$

The above equation is called the Euler (or Euler-Lagrange) equation associated to the functional I.

1.3 Variational Method

Variational method allows us to establish existence theorems for differential equations. Applying variational method to a differential problem means reviewing the solutions like critical points of suitable functionals associated to the problem. The Euler-Lagrange equation provides us the so called weak solutions of the differential poblem. For this reason, we have to give the notions of weak solution and energy functional.

1.3.1 Weak solutions and Energy Functional

Consider an easy linear problem.

Let Ω be a bounded open set in $\mathbb{R}^N$, and let f be a continuous function on $\overline{\Omega} \times \mathbb{R}$. Suppose we want to find a function $u : \overline{\Omega} \to \mathbb{R}$ such that

$$(P) = \begin{cases} -\Delta u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Δ is the Laplace operator. This problem is called the homogeneous Dirichlet problem.

A classical solution of (P) is a function $u \in C^2(\overline{\Omega})$ that satisfies (P) for every $x \in \overline{\Omega}$.

Let $v \in C_c^\infty(\Omega)$, we multiply the equation of (P) by v and integrate over Ω

$$-\int_\Omega \Delta u v dx = \int_\Omega f(x,u) v dx$$

Applying the well-known Green's formula and remember that $u = 0$ on $\partial\Omega$, we obtain that if u is a classical solution, then

$$\int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f(x,u) v dx$$

for every $v \in C_c^\infty(\Omega)$.

The regularity assumptions on u and v can be weakened. In fact, for the integrals to be finite it is enough that $u, v \in L^2(\Omega)$ and so do $\frac{\partial u}{\partial x_i}$ and $\frac{\partial v}{\partial x_i}$ for every i.

This motivates the definition of weak solution.

Definition 1.15

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that $f(x,u)v \in L^1(\Omega)$, for each

$u, v \in H_0^1(\Omega)$, and with the following growth condition

$$ess \sup_{(x,t)\in\Omega\times\mathbb{R}} \frac{|f(x,t)|}{1+|t|^q} < +\infty$$

where $q \in]0, 2^*[$.

A weak solution of problem (P) is a function $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f(x,u) v dx$$

for every $v \in H_0^1(\Omega)$.

Remark 1.12 Notice that classical solutions are also weak solutions, while weak solutions are classical solutions only if $u \in C^2(\Omega)$.

Now, we want to show how weak solutions are related to critical points of functionals. At first, we give a definition of energy functional.

Definition 1.16

Consider the problem (P). The functional $I : H_0^1(\Omega) \to \mathbb{R}$ define as

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega \left(\int_0^{u(x)} f(x,t) dt \right) dx$$

is often called the Energy Functional associated to problem (P). The functional I is differentiable on $H_0^1(\Omega)$ and

$$I'(u)v = \int_\Omega \nabla u \cdot \nabla v dx - \int_\Omega f(x, u(x))v dx$$

for every $v \in H_0^1(\Omega)$.

In the light of the last definition, the connection between weak solutions and critical points is evident: indeed, comparing the definition of weak solution and the last relation, one sees that u is a weak solution of problem (P) if and only if u is a critical point of the functional J. The correspondence between weak solutions and critical point of functionals outlined above is valid, of course, for more general nonlinear problems.

Remark 1.13 The growth condition considered on f is essential to have that the energy functional is well-defined on $H_0^1(\Omega)$. Indeed, if $f(t)$ grows faster that $|t|^{2^*-1}$, then $F(t)$ grows faster than $|t|^{2^*}$; therefore, since $H_0^1(\Omega)$ is not embedded in $L^p(\Omega)$ when $p > 2^*$, the integral of $F(u)$ might diverge for some $u \in H_0^1(\Omega)$.

The functionals $I(u)$ are integral functionals that normally contain a term involving the gradient of u. This type of term is bounded below, because it is nonnegative, but in general it is not bounded above. Therefore, if I contains such a term, it may be perhaps minimized, but probably not maximized. For this reason, critical points of I are often its global minima. The concept of convexity is very relevant for the reaserch of global minima.

Definition 1.17

A functional $I : X \to \mathbb{R}$ on a vector space X is called convex if for every $u, v \in X$ and every real $t \in [0, 1]$, there results

$$I(tu + (1-t)v) \leq tI(u) + (1-t)I(v)$$

The functional is called strictly convex if for every $u, v \in X$, with $u \neq v$, and every real $t \in (0, 1)$ there results

$$I(tu + (1-t)v) < tI(u) + (1-t)I(v)$$

Finally, we say that I is (strictly) concave if $-I$ is (strictly) convex.

Theorem 1.9

Let $I : X \to \mathbb{R}$ be a continuous convex functional on a Banach space X. Then, I is weakly lower semicontinuous. In particular, for every sequence $\{u_k\}_{k\in\mathbb{N}} \subset X$ converging weakly to $u \in X$, one has

$$I(u) \leq \liminf_{k\to\infty} I(u_k)$$

A continuous convex functional need not have a minimum, even if it is bounded below. For this reason, the coercivity is necessary.

Definition 1.18

A functional $I : X \to \mathbb{R}$ on a Banach space X is called coercive if, for every sequence $\{u_k\}_{k\in\mathbb{N}} \subseteq X$, it follows that

$$\lim_{\|u_k\|_X \to +\infty} I(u_k) = +\infty$$

A functional I is called anticoercive if $-I$ is coercive.

The following result is fundamental.

Theorem 1.10

Let X be a reflexive Banach space and let $I : X \to \mathbb{R}$ be a continuous, convex and coercive functional. Then, I has a global minimum point.

Proof.

Put $m = \inf_{u\in X} I(u)$ and let $\{u_k\}_{k\in\mathbb{N}}$ be a minimizing sequence in X. Coercivity implies that $\{u_k\}_{k\in\mathbb{N}}$ is bounded and, since X is reflexive, by the Kakutani and Eberlein-Smulian Theorems, we can extract a subsequence (still denoted u_k) , such that u_k converges weakly to some $u \in X$. Exploiting the theorem (1.9), we then obtain

$$I(u) \leq \liminf_{k \to \infty} I(u_k) = m$$

Therefore, $I(u) = m$ and u is a global minimum for I. $\square$

Our results are devoted, in particular, to the study of least energy solutions and least energy nodal ones. Therefore, we give the following definitions.

Definition 1.19

A weak solution of the problem (P) is said least energy solution if it minimazes the energy functional I on the set of all weak nonzero solutions.

Definition 1.20

A weak solution of the problem (P) is said nodal solution if it changes sing in Ω. Furthermore, it is said least energy nodal solution if it minimazes the energy functional I on the set of all nodal solutions.

Finally, we observe that strict convexity is related to uniqueness properties. Indeed, these two results hold.

Theorem 1.11

Let $I : X \to \mathbb{R}$ be strictly convex. Then, I has at most one minimum point in X.

Theorem 1.12

Let $I : X \to \mathbb{R}$ be strictly convex and differentiable. Then, I has at most one critical point in X.

1.3.2 Spectral properties of Laplacian Operator

Let Ω be a bounded open subset of $\mathbb{R}^N$, and let $q \in L^\infty(\Omega)$.

Definition 1.21

The number $\lambda \in \mathbb{R}$ is an eigenvalue of the operator $\Delta + q(x)$ under Dirichlet boundary conditions if there exists $\varphi \in H_0^1(\Omega)\backslash\{0\}$ such that

$$\begin{cases} -\Delta\varphi + q(x)\varphi = \lambda\varphi & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega \end{cases}$$

The function φ is called an eigenfunction associated to λ.

Remark 1.14

The above problem has to be interpreted in the weak sense, namely φ solves it provided

$$\int_\Omega \nabla\varphi \cdot \nabla v dx + \int_\Omega q(x)\varphi v dx = \lambda \int_\Omega \varphi v dx$$

for all $v \in H_0^1(\Omega)$.

If $q \equiv 0$, the corresponding numbers λ are simply called the eigenvalues of the Laplacian.

The next theorem collects the basic properties of eigenvalues and eigenfunctions.

Theorem 1.13

Let $\Omega \subseteq \mathbb{R}^N$ be bounded and open and let $q \in L^\infty(\Omega)$. Then, there exist two sequences $\{\lambda_k\}_{k\in\mathbb{N}} \subseteq \mathbb{R}$ and $\{\varphi_k\}_{k\in\mathbb{N}} \subseteq H_0^1(\Omega)$ such that

(1) *Each λ_k is an eigenvalue of $\Delta + q(x)$ and each φ_k is an eigenfunction corresponding to λ_k;*

(2) $\lim_{k\to+\infty} \lambda_k = +\infty$*;*

(3) $\{\varphi_k\}_{k\in\mathbb{N}}$ *is an orthonormal basis of $L^2(\Omega)$;*

(4) *For every $k \in \mathbb{N}$, $\varphi_k \in L^\infty(\Omega)$ and $\varphi_k(x) \neq 0$ a.e. in Ω.*

Eigenvalues and eigenfunctions play a very relevant role in nonlinear elliptic problems. The most important eigenvalue is the smallest, λ_1, that is called the first eigenvalue or the principal eigenvalue. Some properties of λ_1 are listed in the next result.

Theorem 1.14 *[Variational characterization of the first eigenvalue]*

Let Ω be an open and bounded subset of $\mathbb{R}^N$ and let $q \in L^\infty(\Omega)$. Define a functional $Q : H_0^1(\Omega)\backslash\{0\} \to \Omega$ as

$$Q(u) = \frac{\displaystyle\int_\Omega |\nabla u|^2 dx + \int_\Omega q(x) u^2 dx}{\displaystyle\int_\Omega u^2 dx}$$

This functional is called the Rayleigh quotient.

Then

(1) $\min_{u \in H_0^1(\Omega)\setminus\{0\}} Q(u) = \lambda_1$;

(2) $Q(u) = \lambda_1$ *if and only if* u *is a weak solution of*

$$(PD) = \begin{cases} -\Delta u + q(x)u = \lambda_1 u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

(3) *Every non identically zero solution of* (PD) *has constant sign in* Ω *(in particular, it is almost everywhere different from zero in* Ω*).*

(4) *The set of solutions of* (PD) *has dimension one; one says that* λ_1 *is simple.*

Remark 1.15 In the case $\lambda_1 > 0$, by conclusion (2) of Theorem 1.14 it is easy to realize that the quantity

$$(u, v) = \int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega q(x)uv dx$$

defines a scalar product on $H_0^1(\Omega)$ that induces a norm equivalent to the usual one.

If we fix an eigenfunction φ_1, the eigenfunctions associated to λ_1 are all of the form $\vartheta\varphi_1$, for $\vartheta \in \mathbb{R}$ and for some $\varphi_1 \in H_0^1(\Omega)$. If Ω is regular, the function $\varphi_1 \in C^1(\overline{\Omega})$. Finally, we observe that the role of eigenvalues is of fundamental importance for the solvability of linear problems.

Remark 1.16 Under regularity conditions, it is possible to prove that nonnegative and nonzero solutions are actually everywhere positive in Ω. This is the content of the strong maximum principle.

1.4 Nehari manifold

Nehari manifold is a manifold of functions, whose definition is motivated by the work of German mathematician Zeev Nehari.

1.4.1 Definition

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain. Consider the following Dirichlet problem

$$(P_1) = \begin{cases} -\Delta u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function and let I be the energy functional associated to the problem (P_1) defined in (1.16) in the Sobolev space $H_0^1(\Omega)$.

When I is of class C^1, the sets

$$\mathcal{N} = \{u \in H_0^1(\Omega) \setminus \{0\} : I'(u)(u) = 0\} \tag{1.4.1}$$

is the so called Nehari manifold associated to I, and

$$\mathcal{N}_{\pm} = \{u \in H_0^1(\Omega) : \; u_+, u_- \in \mathcal{N}\} \tag{1.4.2}$$

is the nodal Nehari manifold.

In the case of nonlinearity of the type $f(x,u) = \lambda |u|^{s-2}u$ with $\lambda > 0$ and $s \in (2, 2^*)$, the functional I restricted to $\mathcal{N}$ is coercive and, since $\mathcal{N}$ is weakly closed in this case, it admits a global minimum $u \in \mathcal{N}$. Hence, there exists a Lagrange multiplier $\rho \in \mathbb{R}$ such that

$$I'(u)(v) + \rho J'(u)(v) = 0$$

for all $v \in H_0^1(\Omega)$, where $J(u) = I'(u)(u)$.

If we put $v = u$ in the above equation, we easily obtain $\rho = 0$ and therefore u is a least energy solution, beacause $S \subset \mathcal{N}$, where S is the set of all nonnegative solutions to problem (P_1).

In order to finding global minimizers of the restriction of I to $\mathcal{N}$ and $\mathcal{N}_{\pm}$ it is convenient to introduce the following sets

$$\mathcal{A} = \{u \in H_0^1(\Omega) \setminus \{0\} : I'(u)(u) \leq 0\} \tag{1.4.3}$$

and

$$\mathcal{A}_{\pm} = \{u \in H_0^1(\Omega) : \; u_+, u_- \in \mathcal{A}\} \tag{1.4.4}$$

In fact, a global minimum point $u^* \in \mathcal{A}$ of $I_{|\mathcal{A}}$ belongs to $\mathcal{N}$, otherwise u^* should belong to the open set $\{u \in H_0^1(\Omega) \setminus \{0\} : I'(u)(u) < 0\} \subset \mathcal{A}$. Therefore, u^* should be a nonzero local minimum of I and belongs to S, which is contained in $\mathcal{N}$.

In the same way, we can prove that a global minimun point $u^* \in \mathcal{A}_\pm$ of $I_{|\mathcal{A}_\pm}$ must belong to $\mathcal{N}_\pm$.

The advatage of considering the sets $\mathcal{A}$ and $\mathcal{A}_\pm$ in place of $\mathcal{N}$ and $\mathcal{N}_\pm$ relies on the fact that, for certain kind of nonlinearites $f(x,t)$, the sets $\mathcal{A}$ and $\mathcal{A}_\pm$ are weakly compact and so minimizers of I on these sets are promptly found via Weierstrass Theorem.

Chapter 2

Main results

In this chapter we present our main results related to the existence of least energy solutions and least energy nodal ones for a Dirichlet problem.

2.1 Basic definitions and main assumptions

Let Ω be a bounded smooth domain in $\mathbb{R}^N$ and we consider the Dirichlet problem

$$\begin{cases} -\Delta u = f(x,u), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{P}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathédory function satisfies the subcritical growth condition, that is for some constant $C > 0$ and some $p \in (1, p^*)$, it holds

$$|f(x,t)| \leq C(1 + |t|^{p-1}) \tag{2.1.1}$$

for almost all $x \in \Omega$ and for all $t \in \mathbb{R}$.

Furthermore, for any $m \in (1, 2^*)$ and with $N \geq 3$, we denote by c_m the constant

$$c_m := \sup_{u \in H_0^1(\Omega)} \frac{\|u\|_m}{\|u\|}$$

which is the best constant for the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^m(\Omega)$.

We remember that

$$\|u\| := \left(\int_\Omega |\nabla u|^2 dx \right)^{\frac{1}{2}}$$

is the norm in $H_0^1(\Omega)$ defined by Theorem 1.3, meanwhile

$$\|u\|_m := \left(\int_\Omega |u|^m dx \right)^{\frac{1}{m}}$$

is the standard norm in $L^m(\Omega)$. Moreover, we will considered the energy functional I defined by

$$I(u) = \int_\Omega |\nabla u|^2 dx - \int_\Omega \left(\int_0^{u(x)} f(x,t) dt \right) dx$$

with $u \in H_0^1(\Omega)$.

Proposition 2.1 *Under the condition (2.1.1), the functional I associated to (P) is sequentially weakly semicontinuous and of class C^1 in $H_0^1(\Omega)$.*

Finally, let $K \subseteq \Omega$ be a set of positive measure and we will considered the following conditions on the nonlinearity f:

(i) $\liminf_{\xi \to 0^+} \frac{\int_0^\xi f(x,t)dt}{\xi^2} \geq \frac{\lambda_1}{2}$, uniformly in $\Omega \setminus K$, and $\liminf_{\xi \to 0^+} \frac{\int_0^\xi f(x,t)dt}{\xi^2} > \frac{\lambda_1}{2}$, uniformly in K.

(ii) $\liminf_{\xi \to 0^+} \frac{\int_0^\xi f(x,t)dt}{\xi^2} > -\infty$, uniformly in $\Omega \setminus K$, and $\limsup_{\xi \to 0^+} \frac{\int_0^\xi f(x,t)dt}{\xi^2} = +\infty$, uniformly in K.

(iii) there exists $\delta > 0$ such that $f(x,\xi)\xi - 2\int_0^\xi f(x,t)dt < 0$, for all $\xi \in [-\delta, \delta] \setminus \{0\}$, and for almost every $x \in \Omega$.

(iv) $\limsup_{t \to 0} \frac{f(x,t)}{t} < \lambda_1$, uniformly in Ω.

(v) $\limsup_{|\xi| \to +\infty} \frac{\int_0^\xi f(x,t)dt}{\xi^2} < \frac{\lambda_1}{2}$, uniformly in $x \in \Omega$;

(vi) there exist $c > 2$ and $M > 0$ such that $0 \leq c \int_0^\xi f(x,t) \leq \xi f(\xi)$, for almost all $x \in \Omega$ and for all $\xi \in \mathbb{R} \setminus (-M, M)$.

2.2 Existence of least energy solutions

Dirichlet problem, like (P), arises in a lot of applications (such as chemical reaction, population dynamics, physical problems). Therefore, solutions have an important concrete meaning and positive solutions often are the only relevant ones. For this reason, positive solutions to nonlinear elliptic equations have been studied intensively in the last fifty years. Here, for the scopes of our treatment, we want to mention the paper [11], where the problem (P) is studied in the case of nonlinearity of the type

$$f(x,u) = \lambda|u|^{s-2}u - |u|^{r-2}u \qquad (*)$$

with $r, s \in (1,2)$ and where the existence of positive solutions is proved for λ sufficientely large.

For the same $\lambda's$, the multiplicity of nonnegative solutions was proved in [3]. In this paper, it is also proved that at least one of the solutions is positive.

In our main results we will prove the existence of least energy solutions and least energy nodal solutions for general nonlinearities $f(x,t)$, satisfying some of the assumptions $(i), (ii), (iii), (iv), (v), (vi)$, which include those as in $(*)$. Our results extends in a more general setting other known results, and they seems new for nonlinearities satisfying certain groups of assumptions (among $(i) - (vi)$), which include the special case of nonlinearity of the type $(*)$.

Our first main result concerns the existence of least energy solutions and states as follows.

Theorem 2.1

Assume that there hold

a) condition (i) or (ii) and condition (v);

b) condition (iii) or (iv) and condition (v) or (vi).

Then, if the set S of all nonzero weak solutions is nonempty, there exists $u_0 \in S$ such that $I(u_0) = \inf_{u \in S} I(u)$.

Proof.

We suppose that the set S is nonempty and let

$$F(x, \xi) = \int_0^{\xi} f(x, t)dt$$

for all $(x, \xi_i) \in \Omega \times \mathbb{R}$. Let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence in S such that

$$\lim_{n \to +\infty} I(u_n) = \inf_S I$$

At first, we prove that under condition (v) or under condition (vi), the sequence $\{u_n\}_{n \in \mathbb{N}}$ is bounded.

We assume the condition (v) and can fix $\rho \in \mathbb{R}$ and $\delta > 0$ such that

$$\xi^{-2} \int_0^{\xi} f(x,t)dt \le \rho < \frac{\lambda_1}{2}$$

for almost every $x \in \Omega$ and for all $\xi \in \mathbb{R} \setminus]-\delta, \delta[$.

Therefore, recalling the growth condition (2.1.1), for some positive constant C_1 and for every $u \in H_0^1(\Omega)$, we have

$$\begin{aligned} I(u) = \ & \frac{1}{2}\|u\|^2 - \int_{\Omega} F(x,u(x))dx = \\ & \frac{1}{2}\|u\|^2 - \int_{|u(x)|\le\delta} F(x,u(x))dx - \int_{|u(x)|>\delta} F(x,u(x))dx \ge \\ & \frac{1}{2}\|u\|^2 - \int_{|u(x)|\le\delta} \left| \int_0^{u(x)} \sup_{|s|\le\delta} |f(x,s)| dt \right| dx - \int_{|u(x)|>\delta} \rho u(x)^2 dx \ge \\ & \frac{1}{2}\|u\|^2 - C_1 - \rho \int_{\Omega} (u(x))^2 dx \ge \frac{1}{2}\|u\|^2 - C_1 - \rho \frac{1}{\lambda_1}\|u\|^2 = \\ & \frac{1}{\lambda_1}\left(\frac{\lambda_1}{2} - \rho\right)\|u\|^2 - C_1. \end{aligned}$$

because

$$\begin{aligned} \int_{|u(x)|\le\delta} F(x,u(x))dx \ & = \int_{|u(x)|\le\delta} \left(\int_0^{u(x)} f(x,t)dt \right) dx \le \\ & \int_{|u(x)|\le\delta|} \left(\int_0^{u(x)} \left| \sup_{|s|\le\delta} |f(x,t)| \right| dt \right) dx \le \\ & \sup_{|s|\le\delta} |f(x,s)| \int_{|u(x)\le\delta|} |u(x)| dx \le \\ & \delta meas(\Omega) \sup_{|s|\le\delta} |f(x,s)| = C_1 \end{aligned}$$

and, by exploiting (2.2.1) and Theorem 1.14,

$$\begin{aligned}\int_{|u(x)|>\delta} F(x,u(x))dx &= \int_{|u(x)|>\delta}\left(\int_0^{u(x)} f(x,t)dt\right)dx \leq \\ &\int_{|u(x)|>\delta|} \rho[u(x)]^2 dx \leq \rho\frac{\|u\|^2}{\lambda_1}\end{aligned}$$

Therefore,

$$\lim_{\|u\|\to+\infty} I(u) = +\infty \tag{2.2.1}$$

This clearly implies the boundedness of $\{u_n\}_{n\in\mathbb{N}}$, because I is bounded from above on bounded set by the subcrititcal growth condition (2.1.1) and Sobolev embeddings.

Now, we suppose the (vi) holds.

Taking also (2.1.1) into account, for some positive constant C_2 and for every $u \in S$, we obtain

$$\begin{aligned}0 = \;& I'(u)(u) = \|u\|^2 - \int_\Omega f(x,u(x))u(x)dx = \\ & \|u\|^2 - \int_{|u(x)|\leq M} f(x,u(x))u(x)dx - \int_{|u(x)|>M} f(x,u(x))u(x)dx \leq \\ & \|u\|^2 - C_2 - c\int_{|u(x)|>M} F(x,u(x))dx.\end{aligned}$$

and for some positive constant C_3, we have

$$\begin{aligned} I(u) = \ & \frac{1}{2}\|u\|^2 - \int_\Omega F(x,u(x))dx = \\ & \frac{1}{2}\|u\|^2 - \int_{|u(x)|\leq M} F(x,u(x))dx - \int_{|u(x)|>M} F(x,u(x))dx \geq \\ & \left(\frac{1}{2} - \frac{1}{c}\right)\|u\|^2 - C_3. \end{aligned}$$

Also in this case we have

$$\lim_{\|u\|\to+\infty} I(u) = +\infty$$

and then the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded.

At this point, we suppose (v) or (vi) holds.

By the reflexivity of $H_0^1(\Omega)$, there exists $u^* \in H_0^1(\Omega)$ such that, up to a subsequence, $u_n \to u^*$ weakly in $H_0^1(\Omega)$ and strongly in $L^m(\Omega)$, for each $m \in (1, p^*)$. To obtain our thesis, we have to prove that $u^* \in S$.

For this purpose, we observe that, being u_n a weak solution for the problem (P), for each $v \in H_0^1(\Omega)$, one has

$$\int_\Omega \nabla u_n(x)v(x)dx = \int_\Omega f(x,u_n(x))v(x)dx \tag{2.2.2}$$

for each $n \in \mathbb{N}$ and, passing to the limit as $n \to +\infty$, we obtain

$$\int_\Omega \nabla u^*(x)v(x)dx = \int_\Omega f(x,u^*(x))v(x)dx.$$

which prove that u^* is a weak solution of problem (P).

Let us to show that u^* is nonzero.

We have already proved that the energy functional I is coercive (i.e. the limit (2.2.1) holds) if the condition $v)$ holds. By a routine argument, we know that I has a global minimizer in $H_0^1(\Omega)$, which is weak solution to problem (P). We first suppose that condition (i) holds.

Let $\varphi_1 \in C^1(\overline{\Omega})$ be the positive eigenfunction associated to λ_1 normalized with respect to the sup-norm. There exist $\delta, \rho > 0$ be such that

$$F(x,\xi) \geq \left(\frac{\lambda_1}{2} + \rho\right)\xi^2 \tag{2.2.3}$$

for all $\xi \in [0, \delta]$ and for almost all $x \in K$, and

$$F(x,\xi) \geq \left(\frac{\lambda_1}{2} - \sigma\rho\right)\xi^2 \tag{2.2.4}$$

for all $\xi \in [0, \delta]$ and for almost all $x \in \Omega \setminus K$, where

$$\sigma = \frac{\displaystyle\int_K \varphi_1(x)^2 dx}{2\left(1 + \displaystyle\int_{\Omega\setminus K} \varphi_1(x)^2 dx\right)}$$

We put $\varphi_\xi = \xi \cdot \varphi_1$, for all $\xi > 0$ and, using the above inequality, for $\xi \in [0, \delta]$, we obtain

$$\begin{aligned}
I(\varphi_\xi) = \; & \frac{\xi^2}{2}\|\varphi_1\|^2 - \int_K F(x, \xi\varphi_1(x))dx - \int_{\Omega\setminus K} F(x, \xi\varphi_1(x))dx \leq \\
& \frac{\xi^2}{2}\|\varphi_1\|^2 - \xi^2\left(\frac{\lambda_1}{2} + \rho\right)\int_K \varphi_1(x)^2 dx - \xi^2\left(\frac{\lambda_1}{2} - \sigma\rho\right)\int_{\Omega\setminus K} \varphi_1(x)^2 dx = \\
& \frac{\xi^2}{2}\left(\|\varphi_1\|^2 - \lambda_1 \int_\Omega \varphi_1(x)^2 dx\right) - \xi^2\rho\left(\int_K \varphi_1(x)^2 dx - \sigma \int_{\Omega\setminus K} \varphi_1(x)^2 dx\right) = \\
& \frac{\xi^2}{2}\left(\|\varphi_1\|^2 - \lambda_1 \int_\Omega \varphi_1(x)^2 dx\right) - \xi^2\rho \int_K \varphi_1(x)^2 dx\left(1 - \frac{\int_{\Omega\setminus K}\varphi_1(x)^2 dx}{2(1 + \int_{\Omega\setminus K}\varphi_1(x)^2 dx)}\right) \leq \\
& \frac{\xi^2}{2}\left(\|\varphi_1\|^2 - \lambda_1 \int_\Omega \varphi_1(x)^2 dx\right) - \frac{\xi^2\rho}{2}\int_K \varphi_1(x)^2 dx = \\
& -\frac{\xi^2\rho}{2}\int_K \varphi_1(x)^2 dx < 0.
\end{aligned}$$

Therefore, $\inf\limits_{H_0^1(\Omega)} I < 0$ and this implies $u^* \neq 0$.

Now, we suppose the condition (ii) holds and we can find $\delta, T > 0$ such that

$$F(x, \xi) \geq -T\xi^2.$$

for all $\xi \in [0, \delta]$ and for almost all $x \in \Omega \setminus K$. We also fix a compact set $K_0 \subseteq K$ with positive measure and an open set Ω_0 such that $K_0 \subset \Omega_0 \subset \overline{\Omega_0} \subset \Omega$ and $meas(\Omega_0 \setminus K_0) < \frac{1}{2T} meas(K_0)$ and let $\varphi \in C^1(\overline{\Omega})$ be a nonnegative function such that $\sup\limits_{x\in\Omega} \varphi(x) = 1$, $\varphi \equiv 1$ in K_0, $\varphi \equiv 0$ in $\Omega \setminus \overline{\Omega_0}$. Moreover, choosing δ smaller if necessary, we get

$$F(x, \xi) \geq R\xi^2$$

for all $\xi \in [0, \delta]$ and for almost all $x \in K$, where $R = \dfrac{\|\varphi\|^2}{meas(K_0)} + 1$.

We put again $\varphi_\xi = \xi\varphi$ for all $\xi > 0$, and obtain, for $\xi \in [0, \delta]$,

$$
\begin{aligned}
I(\xi\varphi) = \; & \frac{\xi^2}{2}\|\varphi\|^2 - \int_{K_0} F(x, \xi\varphi(x))dx - \int_{\Omega\setminus K_0} F(x, \xi\varphi(x))dx \leq \\
& \frac{\xi^2}{2}\|\varphi\|^2 - \int_{K_0} R\xi^2\varphi(x)^2 dx - \int_{\Omega\setminus K_0} T\xi^2\varphi(x)^2 dx = \\
& \frac{\xi^2}{2}\|\varphi\|^2 - R\xi^2 \int_{K_0} \varphi(x)^2 dx - T\xi^2 \int_{\Omega\setminus K_0} \varphi(x)^2 dx = \\
& \frac{\xi^2}{2}\|\varphi\|^2 - R\xi^2 meas(K_0) - T\xi^2 \int_{\Omega\setminus K_0} \varphi(x)^2 dx \leq \\
& \frac{\xi^2}{2}\|\varphi\|^2 - R\xi^2 meas(K_0) + T\xi^2 \int_{\Omega_0\setminus K_0} \sup_{\Omega} \varphi(x)^2 dx = \\
& \frac{\xi^2}{2}\|\varphi\|^2 - R\xi^2 meas(K_0) + T\xi^2 meas(\Omega_0 \setminus K_0) < \\
& \frac{\xi^2}{2}\|\varphi\|^2 - \left(\frac{\|\varphi\|^2}{meas(K_0)} + 1\right)\xi^2 meas(K_0) + T\xi^2 meas(\Omega_0 \setminus K_0) < \\
& \frac{\xi^2}{2}\|\varphi\|^2 - \xi^2\|\varphi\|^2 - \xi^2 meas(K_0) + T\xi^2 \frac{1}{2T} meas(K_0) = \\
& -\frac{\xi^2}{2}\|\varphi\|^2 - \frac{1}{2} meas(K_0) < 0.
\end{aligned}
$$

So $\inf\limits_{H_0^1(\Omega)} I < 0$ and this implies $u^* \neq 0$.

Therefore, the proof in the case of assumption a) is complete.

Now, suppose the assumption (b) holds.

Also in this case, we have to prove that u^* is nonzero.

On the contrary, we assume $u^* = 0$. Recall that for all $n \in \mathbb{N}$ one has

$$
\begin{cases}
-\Delta u_n = f(x, u_n), & \text{in } \Omega \\
u_n = 0 & \text{on } \Omega
\end{cases}
$$

which means that u_n satisfies equation (2.2.2). Testing this equation with $\varphi = u_n$, we get

$$
\|u_n\|^2 = \int_\Omega f(x, u_n(x)) u_n(x) dx. \tag{2.2.5}
$$

Without loss of generality, we can assume $p \in [2, 2^*[$ in the growth condition (2.1.1). Since $u_n \to 0$ strongly in $L^p(\Omega)$, then there exists $g \in L^p(\Omega)$ such that $|u_n(x)| \leq g(x)$, for almost all $x \in \Omega$. We consider

$$
a_n(x) = \frac{f(x, u_n(x))}{1 + |u_n(x)|}
$$

and, because of the increasing of the function $t \to \dfrac{1 + t^{p-1}}{1 + t}$ in $[1, +\infty[$, we can obtain the following estimate

$$
|a_n(x)| \leq C \frac{1 + |u_n(x)|^{p-1}}{1 + |u_n(x)|} \leq a(x) := C \left(1 + \frac{1 + |g(x)|^{p-1}}{1 + |g(x)|} \right)
$$

for almost all $x \in \Omega$.

Furtheremore, one has that $a \in L^{\frac{p}{p-2}}(\Omega)$ and $\dfrac{p}{p-2} > \dfrac{N}{2}$, and that $a_n(x) \to 0$ as

$n \to +\infty$ almost everywhere in Ω, because $u^* = 0$ is a solution of problem (P) if and only if $f(x, 0) = 0$.

By the Dominated Convergence Theorem, it follows

$$\lim_{n \to +\infty} \|a_n\|_{L^{\frac{p}{p-2}}} = 0. \tag{2.2.6}$$

Since u_n is a weak solution of the problem

$$\begin{cases} -\Delta u = a_n(x)(1 + |u|) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

by standard regularity results (see [15]), we have $u_n \in C^{1,\beta}(\overline{\Omega})$, for some $\beta \in (0, 1)$. By Theorem 8.16 of [9], there exists a constant $C > 0$ independent of u_n such that

$$\|u_n\|_\infty \leq C\|a_n\|_{L^{\frac{p}{p-2}}(\Omega)}(1 + \|u_n\|_\infty)$$

Exploiting (2.2.6), we infer

$$\lim_{n \to +\infty} \|u_n\|_\infty = 0. \tag{2.2.7}$$

We know that I is sequentially weakly lower semicontinuous and it follows

$$0 = I(u^*) \leq \liminf_{n \to +\infty} I(u_n) = \inf_S I. \tag{2.2.8}$$

At this point, because of (2.2.7), we can choose $n \in \mathbb{N}$ such that $\|u_n\|_\infty < \delta$. Thanks to (2.2.5), (2.2.8) and (iii), we have

$$\begin{aligned} I(u_n) = & \frac{1}{2}\|u_n\|^2 - \int_\Omega F(x, u_n(x))dx = \\ & \int_\Omega \left(\frac{1}{2}f(x, u_n)u_n(x) - F(x, u_n)\right) dx < \\ & 0 = I(u^*) \le \inf_S I, \end{aligned}$$

which follows from being u_n a non zero function.

This is in contradiction with $u_n \in S$ and so u^* is nonzero. Thus, in the case of assumptions $(iii) - (v)$ or $(iii) - (vi)$ hold, the proof is completed.

Finally, we suppose that assumption (iv) holds.

Let $\eta \in (0, \lambda_1)$ be such that

$$\limsup_{t \to 0^+} \frac{f(x,t)}{t} < \lambda_1 - \eta$$

uniformly in Ω.

Exploiting also (2.1.1), we find a constant $C_1 > 0$ such that

$$|f(x,t)| \le (\lambda_1 - \eta)|t| + C_1|t|^{p-1} \tag{2.2.9}$$

for almost all $x \in \Omega$ and for all $t \in \mathbb{R}$.

We have already prove that, under condition (v) or condition (vi), every minimizing sequence of $I_{|S}$ is bounded. Therefore, if $\{u_n\}$ is a sequence in S which minimizes $I_{|S}$, there exists $u^* \in H_0^1(\Omega)$ such that $u_n \to u^*$ weakly in $H_0^1(\Omega)$ and, as noted above, u^* is a weak solution of problem (P).

Hence,we have to prove that u^* is non zero. To this aim, exploting (2.2.9), we have

$$\begin{aligned} 0 = I'(u_n)(u_n) &= \|u_n\|^2 - \int_\Omega f(x, u_n(x))u_n(x)dx \geq \\ &\geq \|u_n\|^2 - (\lambda_1 - \eta) \int_\Omega |u_n|^2 dx - C_1 \int_\Omega |u_n|^p dx \geq \\ &\geq \|u_n\|^2 - (\lambda_1 - \eta)\frac{1}{\lambda_1}\|u_n\|^2 - C_1 \int_\Omega |u_n|^p dx = \\ &= \frac{\eta}{\lambda_1}\|u\|^2 - C_1 \int_\Omega |u_n|^p dx = \\ &= \frac{\eta}{\lambda_1} c_p^{-2} \left(\int_\Omega |u_n|^p dx \right)^{\frac{2}{p}} - C_1 \int_\Omega |u_n|^p dx, \end{aligned} \tag{2.2.10}$$

for each $n \in \mathbb{N}$. This implies,

$$\int_\Omega |u_n|^p dx \geq \left(\frac{\eta}{\lambda_1 C_1 {c_q}^2} \right)^{\dfrac{p}{p-2}} \tag{2.2.11}$$

Nevertheless, being the functional $u \in H_0^1(\Omega) \to \int_\Omega |u|^p dx$ sequentially weakly continuous, one also has

$$\int_\Omega |u^*|^q dx \geq \left(\frac{\eta}{\lambda_1 C_1 {c_q}^2} \right)^{\dfrac{p}{p-2}}$$

Therefore u^* is nonzero.

The proof of Theorem (2.1)1 is complete. □

2.3 Existence of least energy nodal solutions

We are here interested in studying the nodal solutions of problem (P), which minimize the functional I on the set $S_{\pm}$ of all nodal solutions.

This problem was studied by Castro, Cossio and Neuberger in [6].

They considered the problem (P), where f is superlinear and subcritical in Ω and proved the exixstence of at least one nodal solution, which changes sign exactly once in Ω.

In the case of p-Laplacian, the problem of existence of nodal solutions was studied by Grumiau and Parini in [10], who considered nonlinearity of the type $f(x,u) = \lambda|u|^{s-2}u$, with $s \in (p, p^*)$ and p^* the critical exponent for the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$.

Still today, the existence of nodal solutions to (P) does not seem to be proved in the case that f satisfies the assumption (iii).

The next result gives the existence of nodal solutions also in this case. But, we need to introduce the following additional condition

$(\bar{i})$ there exists $\delta > 0$ such that $f(x,t)t \geq 0$, for almost all $x \in \Omega$ and for all $t \in [-\delta, \delta]$.

Theorem 2.2

Assume b) of Theorem 1 and the additional condition $(\bar{i})$ in the case of (iii) holds. Then, if the set $S_\pm$ of all sign-changing weak solutions is nonempty, there exists $u_0 \in S_\pm$ such that $I(u_0) = \inf_{u \in S_\pm} I(u)$.

Proof.

We suppose that $S_\pm$ is nonempty and consider a sequence $\{u_n\}_{n\in\mathbb{N}}$ in $S_\pm$, which minimizes $I_{|S_\pm}$.

We have already proved in Theorem 2.1 that $\{u_n\}$ is bounded in $H_0^1(\Omega)$ and weakly converges to some $u_* \in H_0^1(\Omega)$, which is a solution of problem (P).

We want to show that u_* is sign-changing.

Suppose, at first, that (iii) and $(\bar{i})$ hold. Arguing as in Theorem 2.1, we deduce that u_* is nonzero. Assume, on the contrary, that u_* is not sign-changing and suppose that u_* is nonnegative (the arguments in the case of u_* nonpositive are similiar).

By standard results, weak solutions to problem (P) are at least of class $C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$. Thanks to condition $(\bar{i})$, we can apply the Strong Maximum Principle of [16] and the classical Hopf's Lemma. Therefore, u_* is strictly positive in Ω, with $\dfrac{\partial u_*}{\partial \nu} < 0$, where ν is the outer unit normal to $\partial\Omega$, and so u_* belongs to the interior P of the positive cone of $C^1(\overline{\Omega})$. We notice that $u_n - u_*$ is solution to the problem

$$\begin{cases} -\Delta u = b_n(x)(1+|u|), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{P_n}$$

where

$$b_n(x) := \frac{f(x,u_n) - f(x,u_*)}{1+|u_n(x) - u_*(x)|}$$

and

$$\lim_{n\to+\infty} b_n(x) = 0$$

for almost all $x \in \Omega$.

Since $u_n - u_* \to 0$ strongly in $L^p(\Omega)$, there exists $g \in L^p(\Omega)$ such that

$$|u_n(x) - u_*(x)| \leq g(x)$$

for all $n \in \mathbb{N}$ and for almost all $x \in \Omega$.

Furthermore, since $u_* \in C^{1+\beta}(\overline{\Omega})$, we have

$$M := \sup_{\overline{\Omega}} |u| < +\infty$$

Exploiting condition (2.1.1) and the inequality

$$|a+b|^{p-1} \leq 2^{p-1}(|a|^{p-1} + |b|^{p-1})$$

we obtain

$$\begin{aligned}|f(x,u_n(x))-f(x,u_*(x))|\leq\ & |f(x,u_n(x))|+|f(x,u_*(x))|\leq\\ & C(2+|u_n(x)|^{p-1}+|u_*(x)|^{p-1})\leq\\ & C(2+2^{p-1}|u_n(x)-u_*(x)|^{p-1}+(2^{p-1}+1)|u_*(x)|^{p-1})\leq\\ & C_1(1+|u_n(x)-u_*(x)|^{p-1})\end{aligned}$$

for almost all $x\in\Omega$, for all $t\in\mathbb{R}$, where

$$C_1:=\max\{2C+(2^{p-1}+1)CM^{p-1},2^{p-1}C\}$$

Thus, because of the nondescreasing of the function $t\to\dfrac{1+t^{p-1}}{1+t}$ in $[1,+\infty[$, we have

$$|b_n(x)|\leq C_1\frac{1+|u_n(x)-u_*(x)|^{p-1}}{1+|u_n(x)-u_*(x)|}\leq C_1\left(1+\frac{1+g(x)^{p-1}}{1+g(x)}\right):=b(x)$$

for all $n\in\mathbb{N}$ and for almost all $x\in\Omega$, with $b\in L^{\frac{p}{p-2}}(\Omega)$.

Hence, arguing as in Theorem 2.1, we deduce

$$\lim_{n\to+\infty}\|u_n-u_*\|_\infty=0$$

and, in particular,

$$\sup_{n\in\mathbb{N}}\|u_n-u_*\|_\infty<+\infty$$

By Theorem 1 of [13], we obtain

$$\sup_{n\in\mathbb{N}} \|u_n - u_*\|_{C^{1,\beta}(\overline{\Omega})} < +\infty$$

for some $\beta \in (0,1)$. Then, by Ascoli-Arzelà Theorem, up to a subsequence, one has

$$\lim_{n\to+\infty} \|u_n - u_*\|_{C^1(\overline{\Omega})} = 0$$

Since u_* belongs to the interior P of the positive cone of $C^1(\overline{\Omega})$, we get $u_n \in P$, for large $n \in \mathbb{N}$.

This is a contradiction with being u_n sign-changing.

The case of u_* nonpositive is analogous. Therefore, u_* must be sign-changing.

Now, we prove our thesis in the case of condition (iv) holds.

Let $u_n^+ := \max\{u_n, 0\}$ and $u_n^- := \max\{-u_n, 0\}$, for each $n \in \mathbb{N}$.

Of course, $u_n^+, u_n^- \in H_0^1(\Omega)$ and, up to a subsequence,

$$u_n^+ \to u_*^+ := \max\{u_*, 0\}$$

and

$$u_n^- \to u_*^- := \max\{-u_*, 0\}$$

almost everywhere in Ω.

Testing equation (2.2.2) with u_n^+ and u_n^- and arguing as in (2.2.10), we obtain the following inequalities

$$\int_\Omega |u_n^+|^p dx \geq \left(\frac{\eta}{\lambda_1 C_1 c_p^2}\right)^{\dfrac{p}{p-2}}$$

and

$$\int_\Omega |u_n^-|^p dx \geq \left(\frac{\eta}{\lambda_1 C_1 c_q^2}\right)^{\dfrac{p}{p-2}}$$

for all $n \in \mathbb{N}$. Passing to the limit as $n \to +\infty$, we get

$$\int_\Omega |u_*^+|^p dx \geq \left(\frac{\eta}{\lambda_1 C_1 c_p^2}\right)^{\dfrac{p}{p-2}}$$

and

$$\int_\Omega |u_*^-|^p dx \geq \left(\frac{\eta}{\lambda_1 C_1 c_q^2}\right)^{\dfrac{p}{p-2}}$$

for all $n \in \mathbb{N}$.

This implies that $u_*^+ \neq 0$ and $u_*^- \neq 0$ and so $u_* \neq 0$.

The proof of Theorem 2.2 is complete. $\square$

Chapter 3

Special case

In this last chapter, we show an application of our main results (proved in Chapter 2) to a nonlinearity of the type

$$f(x,t) = \lambda|u|^{s-2}u - \mu|u|^{r-2}u \tag{3.0.1}$$

with $\lambda > 0$, $\mu \in \mathbb{R}$ and $r, s \in (1,2)$, $r < s$.

Thus, we will consider the following problem

$$\begin{cases} -\Delta u = \lambda|u|^{s-2}u - \mu|u|^{r-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{$P_{\lambda,\mu}$}$$

with $1 < r < s < 2$, $\lambda > 0$ and $\mu \in \mathbb{R}$.

The following proposition allows us to apply our main results to the nonlinearity (3.0.1).

Proposition 3.1

For each $\lambda > 0$, the nonlinearity f defined in (3.0.1) satisfies

(A) *the assumptions (i), (ii), (iii), (v) and $(\bar{i})$, if $\mu \leq 0$.*

(B) *the assumptions (iii), (iv) and (v), if $\mu > 0$.*

Proof.

The conclusion is a direct consequence of elementary calculations. □

Furthermore, for this type of problems we have also proved that the sets S and $S_{\pm}$ are nonempty at least for some values of λ and μ. To this aim, we recall the next results, which say that.

Theorem 3.1 *[Theorem 3.13 of [13]]*

There exists $\mu_0 > 0$ such that $S \neq \emptyset, S_{\pm} \neq \emptyset$, if $\mu \leq \mu_0$.

Here, S and $S_{\pm}$ are, respectively, the set nonzero solutions and the set of nodal solutions to problem $(P_{\lambda,\mu})$.

At this point, by Proposition 3.1 and Theorem 3.1, we have the following result.

Theorem 3.2 *[Theorem 1 of [14]]*

The problem $(P_{\lambda,\mu})$ admits both a least energy solution and least energy nodal one for *each $\lambda > 0$ and $\mu \leq \mu_0$.*

In order to highlight some open problems connected with problem $(P_{\lambda,\mu})$, we now give some further results.

We start by noticing that when $\mu \leq 0$, there exists a sequence of sign-chaning solutions to problem $(P_{\lambda,\mu})$, which strongly converges to the zero function. Therefore, the sets $\mathcal{A}$ and $\mathcal{A}_\pm$ definded in (1.4.3) and (1.4.4) are not weakly closed.

On the contrary, for $\mu > 0$, we can prove the following result.

Theorem 3.3

Let $\lambda, \mu > 0$ and $r, s \in (1, 2)$, with $r < s$.

Then, the sets $\mathcal{A}$ and $\mathcal{A}_\pm$ are weakly compact.

Proof.

Let $\{u_n\}$ be a sequence in $\mathcal{A}$. Then, for each $n \in \mathbb{N}$, one has

$$\|u_n\|^2 \leq \lambda \|u_n\|_{L^s(\Omega)} \leq c_s^s \lambda \|u\|^s$$

Therefore, being $s < 2$, $\{u_n\}$ is bounded in $H_0^1(\Omega)$.

So, up to a subsequence, $\{u_n\}$ weakly converges to some $u^* \in H_0^1(\Omega)$.

Moreover, passing to the limit as $n \to +\infty$ in the inequality

$$\|u_n\|^2 \leq \lambda \|u_n\|_{L^s(\Omega)} - \mu \|u_n\|_{L^r(\Omega)} \tag{3.0.2}$$

we get

$$\|u^*\|^2 \leq \lambda \|u^*\|^s_{L^s(\Omega)} - \mu \|u^*\|^r_{L^r(\Omega)}.$$

To finish, it remains to prove that $u^* \neq 0$.

Indeed, if we fix $m \in (2, 2^*)$, we can find a constant $M > 0$ such that $\lambda t^s - \mu t^\mu \leq M t^m$, for all $t \geq 0$.

Consequently, using (3.0.2), we obtain

$$\|u_n\|^2_{L^m(\Omega)} \leq c_m^2 \|u_n\|^2 \leq c_m^2 (\lambda \|u_n\|^s_{L^s(\Omega)} - \mu \|u_n\|^r_{L^r(\Omega)}) \leq c_m^2 M \|u_n\|^m_{L^m(\Omega)}$$

from which

$$\|u_n\|_{L^m(\Omega)} \geq \left(\frac{1}{M c_m^2} \right)^{\frac{1}{q-2}}$$

Passing to the limit as $n \to +\infty$ in this last inequality and taking into account the sequential weak continuity of the functional $u \in H^1_0(\Omega) \to \|u\|^m_{L^m(\Omega)}$, we get

$$\|u^*\| \geq \left(\frac{1}{M c_m^2} \right)^{\frac{1}{m-2}}$$

Therefore, $u^* \neq 0$. The weakly compactness of $\mathcal{A}_\pm$ follows being $\mathcal{A}_\pm$ a weakly closed subset of $\mathcal{A}$. $\square$

It is well-known that the energy functional

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{s}\|u\|_s^s + \frac{\mu}{r}\|u\|_r^r \qquad with \qquad u \in H_0^1(\Omega)$$

associated with the problem $(P_{\lambda,\mu})$ is sequentially weakly lower semicontinuous in $\mathcal{A}$ and $\mathcal{A}_\pm$ and so, being $\mathcal{A}$ and $\mathcal{A}_\pm$ nonempty, I has global minima in $\mathcal{A}$ and $\mathcal{A}_\pm$. From the previous discussion about the sets $\mathcal{A}$ and $\mathcal{A}_\pm$, it is quite natural to ask if the global minima of $I_{|\mathcal{A}}$ and $I_{|\mathcal{A}_\pm}$ are in the interior of $\mathcal{A}$ (respectively $\mathcal{A}_\pm$) or in $\mathcal{N}$ (respectively $\mathcal{N}_\pm$). The next result gives an answer to this question.

Corollary 3.1

Let λ, μ, r, s be as in Theorem 3.3 and let $u_0 \in \mathcal{A}$ and $v_0 \in \mathcal{A}_\pm$ be global minima of $I_{|\mathcal{A}}$ and $I_{|\mathcal{A}_\pm}$ (respectively).

Then, $u_0 \in \mathcal{N}$ and $v_0 \in \mathcal{N}_\pm$ and in particular

$$I(u_0) = \inf_{\mathcal{N}} I$$

$$I(v_0) = \inf_{\mathcal{N}_\pm} I.$$

Proof.

Let $u_0 \in \mathcal{A}$ and $v_0 \in \mathcal{A}_\pm$, such that

$$I(u_0) = \inf_{\mathcal{A}} I,$$

$$I(v_0) = \inf_{\mathcal{A}_\pm} I.$$

If $\|u_0\|^2 - \lambda\|u_0\|_s^s + \mu\|u_0\|_r^r < 0$, then u_0 should be a local minimum of I.

Thus, $I'(u_0)(u_0) = \|u_0\|^2 - \lambda\|u_0\|_s^s + \mu\|u_0\|_r^r = 0$ and this is a contradiction.

Therefore, $u_0 \in \mathcal{N}$.

The same argument shows that $v_0 \in \mathcal{N}_\pm$. □

Corollary 3.1 ives rise to the following open question.

(1) Let λ, μ, r, s be as in Theorem 3.3 and let u_0,v_0 be as in Corollary 3.1. From Theorem 3.13 of [11] and Theorem 1 of [12], we know that S and $S_\pm$ are nonempty when the nonlinearity f is of the type

$$f(x,t) = \lambda|t|^{s-2}t - \mu|t|^{r-2}t$$

In the superlinear case, u_0 and v_0 are critical points of I and the proof is based on Deformation Lemma and Miranda Theorem (see [10]).

In our case, u_0 and v_0 are not critical points and so we cannot say if they belong to S and $S_\pm$ (respectively).

If this problem was solved, we could give a variational characterization of u_0 and v_0 in terms of λ, μ and the critical exponent for the Sobolev embedding.

Other open questions are described below.

(2) We wonder if Theorem 3.13 of [11] could be proved also in the quasilinear case and with the p-Laplacian. This extension is very interesting, because the p-Laplacian is degenerate, $I^{'}(u)(u) = 0$ is not a manifold and Implicit Function Theorem cannot apply. However, we are not able to give any reference and prove for this.

(3) Let $\lambda > 0$ be fixed. From Theorem 3.13 of [11] we can obtain, by scaling argument, the existence of $\mu_\lambda > 0$ such that problem $(P_{\lambda,\mu})$ has a positive solution for all $\mu \in]0, \mu_\lambda[$ and no solution for $\mu > \mu_\lambda$. Moreover, Theorem 1.3 of [12] proves the existence of nodal solutions for $\mu \in]0, \mu_\lambda[$.

At this point it is natural to wonder if it is possible to prove a similiar result in the case of the p-Laplacian.

(4) The Theorem 3.13 of [11] proves that problem $(P_{\lambda,\mu})$ has positive solutions also if $0 < r < s < 2$. But, if $r < 1$, the energy functional I associated to $(P_{\lambda,\mu})$ is not

differentiable, even if it has still good properties.

For this reason, the prove of Corollary 3.1 fails and we are not able to prove that the global minima $u_0 \in \mathcal{A}$ and $v_0 \in \mathcal{A}_\pm$ of I belong to $\mathcal{N}$ and $\mathcal{N}_\pm$ (respectively).

In reality, I is differentiable in the interior P of the positive cone of $C^1(\overline{\Omega})$ and, therefore, the prove of Corollary 3.1 would be valid (at least for u_0) if we proved that u_0 belongs to P.

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